VDW’s Thm

Def Let $W, k, c \in \mathbb{N}$. Let $\text{COL} : [W] \to [c]$. A \textbf{mono $k$-AP} is an arithmetic progression of length $k$ where every element has the same color. We often say

$$a, a + d, \ldots, a + (k - 1)d \text{ are all the same color}$$
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$W(1, c) = 1$. 

$W(2, c) = c + 1$. 

By Pigeon Hole Principle. 

$W(k, 1) = k$. 

The mono $k$-AP is $1, 2, \ldots, k$. 

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The next two slides are about what happens

1. Within one block.
2. If we take enough blocks, how they relate.
Within a Block

**Def:** $a, a + d, a + 2d$ is an almost mono 3AP if $\text{COL}(a) = \text{COL}(a + d) \neq \text{COL}(a + 2d)$. The color of an almost mono 3AP is $\text{COL}(a) = \text{COL}(a + d)$. 
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Look at the first three elements of a block of 5:

1. **RRR** or **BBB**. 1-2-3 is mono 3AP.
2. **RBR** or **BRB**. 1-3-5 is mono 3AP or almost mono 3AP.
3. **RBB** or **BRR**. 2-3-4 is mono 3AP or almost mono 3AP.
4. **BBR** or **RRB**. 1-2-3 is almost mono 3AP.
5. **BRB**. 1-3-5 is a mono 3AP or an almost mono 3AP.
6. **BRR**. 2-3-4 is mono 3AP or almost mono 3AP.
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So always get a mono 3AP or an almost mono 3AP. Can assume its almost mono 3AP and its **R**.
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If have A Lot of Blocks Then . . .

We take enough blocks so that for all 2-colorings:

- Two of the blocks are the same color, say $B_i$ and $B_j$.
- $\exists k$ $B_i - B_j - B_k$ is either mono 3AP or almost mono 3AP.

If there are 33 blocks then 2 are the same color. Worst Case $B_1$ and $B_{33}$ same color. So need $B_{65}$ to exist. Hence need to take $W = 5 \times 65 = 365$.

We can get by with LESS blocks - we will consider this point after the proof.
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$W(3, 2) \leq 365$


$\begin{array}{cccc}
R & R & B & d \\
R & R & B & d \\
D & D & d & d \\
\end{array}$

If $?$ is B then get B 3-AP.
If $?$ is R then get R 3-AP.
Done!
**W(3, 2) ≤ 365**

Let \( COL: [W] \rightarrow [2] \).

Break \([W]\) into 65 blocks of size 5 which we think of as being 32-colored.
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- In each block there is a mono 3AP or an almost mono 3AP. (This is why blocks-of-5.)
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```
<table>
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<th>R</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
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<table>
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\[
\begin{array}{cccccc}
 & & d & & d & D \\
 & d & & d & & D \\
R & R & B & & & \\
\end{array}
\begin{array}{cccccc}
 & & d & & d & D \\
 & d & & d & & D \\
R & R & B & & & \\
\end{array}
\begin{array}{cccc}
 & & & \\
 & & d & d \\
\end{array}
\]

If ? is \( B \) then get \( B \) 3-AP.
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Done!
Side Note: Can Get By With Less Blocks

**Warning** This Slide is NOT important.
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If a block is colored **RRRBB** we are done.
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If a block is colored \texttt{RRRBBB} we are done.

So we don’t really have to look at 32 colorings.
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How many colorings of a block already have a mono 3AP.
RRRXY with $X, Y \in \{R, B\}$. 4 colorings.

BBBXY with $X, Y \in \{R, B\}$. 4 colorings.

RBRRR

RBRBR

BRBBB

BRBRB

RBBBBX with $X \in \{R, B\}$. 2 colorings.

BRRRX with $X \in \{R, B\}$. 2 colorings.

RRBBB

BBRRR

There are 16 blocks which already have a mono 3AP. Hence can use $32 - 16 = 16$ blocks. I really do not care.
Side Note: Can Get By With Less Blocks (cont)

$RRRXY$ with $X, Y \in \{R, B\}$. 4 colorings.

$BBBXY$ with $X, Y \in \{R, B\}$. 4 colorings.

$RBRRR$

$RBRBR$

$BRBBB$

$BRBRB$

$RBBBBX$ with $X \in \{R, B\}$. 2 colorings.

$BRRRX$ with $X \in \{R, B\}$. 2 colorings.

$RRBBB$

$BBRRR$

There are 16 blocks which already have a mono 3AP. Hence can use $32 - 16 = 16$ blocks.
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I really do not care.
Is $W(3, 2) = 365$?

No

What is $W(3, 2)$?

One can work out by hand that $W(3, 2) = 9$.

We will later say which VDW numbers are known and how they compare to the bounds given by the proof of VDW's Thm.

**Spoiler Alert** The few known VDW numbers are much smaller than the bounds given by the proof of VDW's Thm.
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$W(3, 3)$

$\text{COL} : \mathcal{W} \rightarrow [3]$. 

Darn. Now what? Discuss. We have 2 almost mono 3APs of different colors that share the same last element.
$W(3, 3)$


How big should the blocks be?
$W(3, 3)$


How big should the blocks be? 7.
$W(3, 3)$

\[ \text{COL: } [W] \rightarrow [3]. \]

How big should the blocks be? 7.

Then $\forall$ 3-coloring of block $\exists$ mono 3AP or almost mono 3AP.
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**The 3-coloring of $[W]$ is a 3$^7$-coloring of the $B_i$’s**

Need for all 3$^7$ colorings of blocks get a mono 3AP or an almost mono 3AP.

Need $2 \times 3^+1$ blocks.

How big should the blocks be? 7.
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**The 3-coloring of \([W]\) is a \(3^7\)-coloring of the \(B_i\)'s**

Need for all \(3^7\) colorings of blocks get a mono 3AP or an almost mono 3AP.

Need \(2 \times 3^1 + 1\) blocks.

Darn. Now what? Discuss
**W(3, 3)**

**COL:** \([W] \rightarrow [3]\).

How big should the blocks be? 7.
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I Like Big Blocks and I Cannot Lie!

Let $W$ be LOTS of blocks of size $7 \times 2 \times (3^7 + 1)$. 
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Let $W$ be LOTS of blocks of size $7 \times 2 \times (3^7 + 1)$. For any 2-coloring of $[W]$ the following happens:
I Like Big Blocks and I Cannot Lie!

Let $W$ be LOTS of blocks of size $7 \times 2 \times (3^7 + 1)$. For any 2-coloring of $[W]$ the following happens:

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Done!
I Like Big Blocks and I Cannot Lie!

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Done!
From what you have seen:

$W(3, c)$
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$W(3, c)$

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- You **COULD** do a proof that $W(3, 4)$ exists. You would need to iterate what I did twice.
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- You can **BELIEVE** that $W(3, c)$ exists though might wonder how to prove it formally.
- There are ways to formalize the proof; however, they are not enlightening.
- The Hales-Jewitt Thm is a general theorem from which VDW is a corollary. We won’t be doing that.
What Did We Use to Prove $W(3, c)$?

$W(2, c) = c + 1$ is just PHP.
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$W(2, 2^5) \implies W(3, 2)$
What Did We Use to Prove $W(3, c)$?

$W(2, c) = c + 1$ is just PHP.

$W(2, 2^5) \implies W(3, 2)$

$W(2, 3^{2 \times 3^7} + 1) \implies W(3, 3)$. 

What Did We Use to Prove $W(3, c)$?

$W(2, c) = c + 1$ is just PHP.

$W(2, 2^5) \implies W(3, 2)$

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$W(2, X) \implies W(3, 4)$ where $X$ is very large.
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Note that we do not do $W(3, 2) \implies W(3, 3)$. 


\( W(4, 2) \)

\[ \text{COL: } [W] \rightarrow [4]. \]
$W(4, 2)$


**Key** Take blocks of size $2W(3, 2)$.
$W(4, 2)$

**COL**: $[\mathcal{W}] \rightarrow [4]$.  

**Key** Take blocks of size $2W(3, 2)$.  
Within a block is mono 4AP or almost mono 4AP.
\( W(4, 2) \)

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**W(4, 2)**

**COL**: $[\mathcal{W}] \to [4]$.

**Key** Take blocks of size $2\mathcal{W}(3, 2)$.
Within a block is mono 4AP or almost mono 4AP.

**Key** Take blocks of size $2\mathcal{W}(3, 2)$.
How many blocks?
$W(4, 2)$

**COL**: \([W] \rightarrow [4]\).

**Key** Take blocks of size $2W(3, 2)$.

Within a block is mono 4AP or almost mono 4AP.

**Key** Take blocks of size $2W(3, 2)$.

How many blocks? Want mono 3AP or almost mono 3AP of blocks.
$W(4, 2)$


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R  R  R  B
d  d  d
R  R  R  B
d  d  d
R  R  R  B
d  d  d
?  d  d  d
D  D  D
\[ W(4, 2) \]

**COL**: \([W] \rightarrow [4]\).

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\[
\begin{array}{cccc}
\text{R} & \text{R} & \text{R} & \text{B} & \cdots & \text{R} & \text{R} & \text{R} & \text{B} & \cdots & \text{R} & \text{R} & \text{R} & \text{B} & \cdots & \text{?} \\
\text{d} & \text{d} & \text{d} & \text{D} & \text{d} & \text{d} & \text{d} & \text{D} & \text{d} & \text{d} & \text{d} & \text{D} & \text{d} & \text{d} & \text{d} & \text{D} & \text{d} & \text{d} & \text{d} & \text{?} \\
\end{array}
\]

If ? is B get mono 4AP.

If ? is R get mono 4AP.

Done!
$W(4, 2)$


**Key** Take blocks of size $2W(3, 2)$. Within a block is mono $4$AP or almost mono $4$AP.

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How many blocks? Want mono $3$AP or almost mono $3$AP of blocks. $2W(3, 2^W(3, 2))$.

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If \(?\) is B get mono 4AP.
If \(?\) is R get mono 4AP.
Done!
You COULD do a proof that $W(k, c)$. You would need to iterate what I did a lot.

You can BELIEVE that $W(k, c)$ exists though might wonder how to prove it formally.

There are ways to formalize the proof; however, the are not enlightening.

The Hales-Jewitt Thm is a general theorem from which VDW is a corollary. We won't be doing that.
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The Hales-Jewitt Thm is a general theorem from which VDW is a corollary. We won’t be doing that.
Induction, But On What?

\[(2, 2) \prec (2, 3) \prec \cdots \prec (3, 2) \prec (3, 3) \prec \cdots \prec (4, 2) \cdots\]

This is an $\omega^2$ induction. The ordering is well-founded so you can do induction. This is an $\omega^2$ induction. That's why the numbers are so large. How large? That takes another entire slide-deck to explain. (Unless you've already seen my slide packet on Primitive Recursive Functions, in which case just know that the proof given gives bounds that are NOT prim rec.)
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