Well Quasi Orders

Exposition by William Gasarch-U of MD

Our Motivating Question

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$L \subseteq \{a, b\}^*$ is often called **a language**.

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Example

$$\mathrm{SUBSEQ}(aaba) = \{e, a, b, aa, ab, ba, aaa, aab, aba, aaba\}.$$

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Question L decidable $\implies$ $\mathrm{SUBSEQ}(L)$ decidable?

Quasi Orders

Def $(X, \preceq)$ is a **Quasi Order** if

- If $x \preceq y$ and $y \preceq z$ then $x \preceq z$ (transitive).
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Most wqo are also partial order, but NOT the one on the HW which caused this hot mess.

Well Quasi Orders

Def $(X, \preceq)$ is a **Well Quasi Order (wqo)** if $(X \preceq)$ is a quasi order AND the following holds:
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there exists $i < j$ with $x_i \preceq x_j$. We call this an **uptick**.

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$$\mathrm{COL}(i,j) = \begin{cases} UP & \text{if } x_i \preceq x_j \\ DOWN & \text{if } x_j \prec x_i \\ INCOMP & \text{if } x_i \text{ and } x_j \text{ are incomparable} \end{cases} \quad (1)$$

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CANT be color INCOMP: Get a sequence with no uptick.
HAS to be color UP- so we get an infinite increasing subsequence.

Now Two Defs of wqo

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Use Def One when want to prove $(X, \preceq)$ is a wqo.

Use Def Two when you already know $(X, \preceq)$ is a wqo.

Interesting Example of a wqo

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Discuss Prove this is a wqo.

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SOME IF IT IS ON THIS SLIDES.
ITS ALSO ALL IN THE NOTES.

Very Hard Theorem (We Won't Prove it)

Def H is a **minor** of G (Denoted by $H \preceq_m G$) if one can obtain H by taking G and carrying out the following operations in some order:

1) Remove a vertex (and all of the edges from it).
2) Remove an edge.
3) Contract an Edge (so merge vertices at ends).

Let $\mathcal{G}$ be the set of all graphs.

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We use $(\mathcal{G}, \preceq_m)$ as an example of a wqo in the next few slides.

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(Planar graphs are closed downward.)

2) O is an **Obstruction Set** for Y if

$$(\forall x \notin Y)(\exists o \in O)[o \preceq_m x].$$

(Obstruction set for Planar graphs is $\{K_{3,3}, K_5\}$.)

Obstruction Set Theorem

Thm Let $(X, \preceq)$ be a wqo. Let $Y \subseteq X$ be closed downward. Then there exists a **Finite Obstruction Set** for Y.

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$$O = \{x \in X - Y \colon (\forall y)[y \prec x \implies y \in Y]\}$$

We claim O is a finite obstruction set.

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1) O is Obstruction: If $z_1 \in X - Y$ then either $z_1 \in O$ (DONE) or $z_1 \notin O$, so there exists $z_2 \in X - Y$ with $z_2 \prec z_1$. Repeat process with z_2. end up with

$$z_1 \succ z_2 \succ z_3 \cdots$$

Has to stop or else have infinite descending sequence. Ends at an element of O.

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2) O is finite: All elements of O are incomparable to each other. If O was infinite then would have an infinite antichain.