**The First Anti-Ramsey Theorem**
**An Exposition by**
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# 1 Introduction

**Definition 1.1** If $n \in \mathsf{N}$ then $[n] = \{1, \ldots, n\}$.

The following is Schur's Theorem which is an early result (1916) in Ramsey theory:

**Theorem 1.2** *For all* $c \in \mathsf{N}$ *there exists* $n \in \mathsf{N}$ *such that, for all* $\mathrm{COL}\colon [n] \to [c]$ *there exists* $x, y, z$ *all the same color such that* $x + y = z$.

In Schur's theorem the goal is a solution to $x + y = z$ where $x, y, z$ are *the same* color. What if you want a solution where $x, y, z$ are *different colors*?

**Definition 1.3** Let $c, n \in \mathsf{N}$ and $\mathrm{COL}\colon [n] \to [c]$. A triple $x, y, z \in [n]$ is *rainbow* if $x, y, z$ are all different colors.

If COL colors all the numbers R then there is no rainbow solution. Our main theorem deals with this issue.

# 2 The Main Theorem

The following theorem was proven in 1987 by V. Alekseev and S. Savchev. The paper is in Russian and is here:
`https://www.kvant.digital/problems/m1040/`

**Theorem 2.1** *Let* $n \in \mathsf{N}$. *Let* $\mathrm{COL}\colon [3n] \to [3]$. *Assume that every color appears in the image* $n$ *times. Then there exists rainbow* $x, y, z \in [3n]$ *such* $x + y = z$

**Proof:**
We let the colors be R,B,G. We can assume COL(1) = R. Let $k$ be such that

$$\mathrm{COL}(1) = \cdots = \mathrm{COL}(k-1) \neq \mathrm{COL}(k).$$

We can assume $\mathrm{COL}(k) = \mathrm{B}$.

We do an example: $n = 7$, $k = 5$, and COL(13) = G. (When we generalize the example we will have $a$ instead of 13). So we have:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | R | R | R | B | | | | | | | | G | | | | | | | | |

We will show that either we get our rainbow solution or COL(12) = R. When we generalize this example (1) we will have $a - 1$ instead of 12, and (2) we will see how getting $\mathrm{COL}(a-1) = R$ helps prove the theorem.

**Case 1:** COL(12) = B. So we have:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | R | R | R | B | | | | | | | B | G | | | | | | | | |

Then $(1, 12, 13)$ is a rainbow solution. (In generalization, 12 is replaced with $a - 1$.)

**Case 2:** COL(12) = G. So we have:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | R | R | R | B | | | | | | | G | G | | | | | | | | |

**Case 2.1:** COL(8) = R. So we have:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | R | R | R | B | | | R | | | | G | G | | | | | | | | |

Then $(5, 8, 13)$ is a rainbow solution. (This case did not use COL(12) = G.) (In generalization, 8 is replaced with $a - k$.)

**Case 2.2:** COL(8) = B. So we have:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | R | R | R | B | | | B | | | | G | G | | | | | | | | |

Then $(4, 8, 12)$ is a rainbow solution.

**Case 2.3:** COL(8) = G. (We will go through cases that look odd; however, they are similar to what happens when we generalize this example.) So we have:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | R | R | R | B | | | G | | | | G | G | | | | | | | | |

**Case 2.3.1:** COL(3) = B. NOT TRUE. (When we generalize this example we will have $a - 2k$ instead of 3.)

**Case 2.3.2:** COL(3) = R. TRUE. Rainbow solution $(3, 5, 8)$.

**Case 2.3.2.1:** COL(3) = G. NOT TRUE.

**Case 3:** COL(12) = R. This must be what happens.

We generalize this example.

**Case 1:** $\exists a \geq k + 1$, COL($a$) = G and COL($a - 1$) = B. Then $(1, a - 1, a)$ is a rainbow solution.

**Case 2:** $\exists a \geq k + 1$, COL($a$) = G and COL($a - 1$) = G.

**Case 2.1:** COL($a - k$) = R. ($a - k \in [n]$ since $a \geq k + 1$.)

Then $(k, a - k, a)$ is a rainbow solution. (This case did not use that COL($a - 1$) = G. )

**Case 2.2:** COL($a - k$) = B.

Then $(k - 1, a - k, a - 1)$ is a rainbow solution.

**Case 2.3:** COL($a-k$) = G. Replicate the reasoning from Case 1,2,2.1,2.2 with $a$ replaced by $a-k$. There is either a rainbow solution or COL($a - 2k$) = G. If COL($a - 2k$) = G then (1) $a - 2k \in [k]$: contradiction, or (2) repeat the argument again. Keep doing this. Eventually there is a rainbow solution since otherwise $\exists i$, $a - ik \in [k]$ and hence cannot be green.

**Case 3:** $\forall a$ if COL($a$) = G then COL($a - 1$) = R. We map every green number $a$ to $a - 1$. This is an injection and everything in the image is red. The number 1 is not in the image since either COL(2) = R or COL(2) = B. Hence there is at least one more green number than red number. But the number of green numbers is the same as the number of red numbers (both numbers are $n$). Hence this case cannot occur.

$\blacksquare$

## 3 What Else is Known?

Theorem 2.1 raises some questions:

1. View the hypothesis Theorem 2.1 each colors appears $\geq n$ times. Can we loosen the requirement? Yes. Schonheim [2] showed

2. What if we have a different equation? Fox, Mahdian, and Radoicic [1] showed the following:

   *For all $n$, for all* $\mathrm{COL}\colon [n] \rightarrow [4]$ *where every color appears* $\geq \frac{n+1}{6}$ *times, there is a rainbow solution to* $x + y = z + w$*. The lower bound on the number of times a color can appear is tight.*

We are sure that other equations have been studied. We will add those laster.

## References


[1] Jacob Fox, Mohammad Mahdian, and Rados Radoicic. Raibnow solutio to the Sidon equation. *Discrete Mathematics*, 308:4773–4778, 2008. `https://mfeapp.baruch.cuny.edu/math/wp-content/uploads/2015/10/20.pdf`.

[2] J. Schonehiem. On partitions of the positive integers with no $x, y, z$ belonging to distinct classes satisfying $x + y = z$. In *Proceedings of the First Conference of the Canadian Number Theory Association*, 1988. `https://www.cs.umd.edu/~gasarch/BLOGPAPERS/rainbow-xyz.pdf`.