

Homework 01, Morally Due Tue Feb 3, 2026

1. (0 points) But DO IT anyway.
 - (a) What is your name? Write it clearly.
 - (b) When is the midterm scheduled (give Date and Time)? If you cannot make it in that day/time see me ASAP.
 - (c) What is the Dead Cat Policy?

2. (30 point)

(a) (10 points) Let $m \in \mathbf{N}$ with $m \geq 2$.

Let $\text{COL}: \binom{\mathbf{N}}{2} \rightarrow [m]$ be

$$\text{COL}(x, y) = x + y \pmod{m}$$

For $0 \leq i \leq m - 1$ let

$$A_i = \{x: x \equiv i \pmod{m}\}$$

Every infinite subset of an A_i is an infinite homog set.

Fill in the XXX in the following statement:

The following are equivalent

- *There is an infinite homog set that is NOT an infinite subset of one of the A_i 's.*
- *m has property XXX.*

(b) (10 points) Let $\text{COL}: \binom{\mathbf{N}}{2} \rightarrow \mathbf{N}$ be

$$\text{COL}(x, y) = |x - y|.$$

Find a number $X \in \mathbf{N}$ such that the following is true, and prove it:

There is a homog set of size X but not of size $X + 1$.

(c) (10 points) Let $\text{COL}: \binom{\mathbf{R} \times \mathbf{R}}{2} \rightarrow \mathbf{R}$ be

$$\text{COL}(x, y) = d(x, y).$$

(x and y are points in the plane and $d(x, y)$ is the distance between x and y .)

So COL takes two points in the plane and returns the length of the line between them.

Find a number $X \in \mathbf{N}$ such that the following is true, and prove it:

There is a homog set of size X but not of size $X + 1$.

3. (30 points) Let x_1, x_2, \dots be a sequence of reals.

Notation: When we write $\text{COL}(i, j)$ we assume $i < j$.

Let $\text{COL}: \binom{\mathbb{N}}{2} \rightarrow [3]$ be defined by

$$\text{COL}(i, j) = \begin{cases} 1 & x_i < x_j; \\ 2 & x_i = x_j; \\ 3 & x_i > x_j. \end{cases} \quad (1)$$

(a) (30 points) Fill in the XXX in the following sentence to make it true.

Apply Ramsey's Theorem to COL. This gives a theorem: given any sequence of reals x_1, x_2, x_3, \dots there exists a subsequence (the homog set) such that XXX.

(b) (0 points) Think about. Let x_1, x_2, \dots be a sequence of reals between 0 and 1. Using Part 1 (so using XXX) what well known theorem can you prove?

4. (20 points) Let $f: Z \rightarrow Z$. Show that there exist an infinite $D \subseteq Z$ such that f restricted to D (with co-domain still Z) is NOT onto.

5. (20 points) (This is not a question in Ramsey Theory but it is a prelude to one of our “Applications.”)

Notation \mathbb{Z} is the integers. $\mathbb{Z}[x_1, \dots, x_n]$ is the set of all polynomials in $\{x_1, \dots, x_n\}$ with coefficients in \mathbb{Z} .

For $p(x, y) \in \mathbb{Z}[x, y]$, we view p as a function from $\mathbb{Z} \times \mathbb{Z}$ into \mathbb{Z} .

Note that $p(x, y)$ might be onto (e.g., $p(x, y) = x + y + 1$) or not (e.g., $p(x, y) = x^2 + y^2 + 10$).

Prove the following:

- (a) FOR ALL $p(x, y) \in \mathbb{Z}[x, y]$ there is an infinite set $D \subseteq \mathbb{Z}$ such that p restricted to $D \times D$ is NOT onto \mathbb{Z} (that is, there is some element of \mathbb{Z} not in the image).
- (b) FOR ALL $p(x, y) \in \mathbb{Z}[x, y]$, if we let $q(x, y) = \lceil p(x, y)^{1/11} \rceil$ (Note that a number has many 11th roots, but only one real one. Take the real one.) then there is an infinite set $D \subseteq \mathbb{Z}$ such that q restricted to $D \times D$ is NOT onto \mathbb{Z} .

6. (EXTRA CREDIT) Prove or Disprove: FOR ALL function $p: Z \times Z$ there is an infinite set $D \subseteq Z$ such that p restricted to $D \times D$ is NOT onto Z (that is, there is some element of Z not in the image).