Homework 11, Morally Due 12:30PM, Tue Apr 28 2026

1. (0 points) What is your name?

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2. (35 points)

(a) (10 points) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ with a combinatorial proof. That is, show that the Right Hand Side solves the problem *how many ways can I pick k Ramsey Theorists from a set of n Ramsey Theorists.*

(You will use this in the next part.)

(b) (15 points) Show that, for all $a, b \geq 2$,
$R(a, b) \leq \binom{a+b-2}{a-1}$.

(c) (10 points) Use the result $R(a, b) \leq \binom{a+b-2}{a-1}$ to get an asymptotic upper bound on $R(k)$ that is LESS THAN 2^{2k-1}.

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3. (30 points)

Notation $\mathsf{N}^{\geq 2}$ means the set $\{2, 3, 4, \ldots\}$.

Def Let $G = (V, E)$ be a graph. $D \subseteq V$ is a *dominating set (DS)* if

$$(\forall u \in V)[u \in D \vee (\exists v \in D)[(u, v) \in E].$$

Every graph has a DS of size n: $D = V$. We do better!

You will prove: *There exists a function* $\alpha\colon \mathsf{N}^{\geq 2} \to (0, 1)$ *such that,*

a) For every $d \in\in \mathsf{N}^{\geq 2}$, $\alpha(d + 1) < \alpha(d)$ *(so* $\alpha(d)$ *is* ***STRICTLY DECREASING****).*

b) For every graph with min degree $\geq d$ *there is a dominating set of size* $\leq \alpha(d)n$.

On the next page we will state the theorem with the function α and sketch the proof. YOU will fill in the details and find a function α that works. We guide this with a series of embedded questions.

Thm There exists a function $\alpha\colon \mathsf{N}^{\geq 2} \rightarrow (0,1)$ such that the following hold:

- α is strictly DECREASING.
- If $G = (V, E)$ is a graph on n vertices with min degree $\geq d$ then G has a dominating set of size $\leq \alpha(d)n$.

Proof Sketch Let p be a probability to be determined by YOU later.

Pick $X \subseteq V$ as follows: For every $v \in V$ choose v with probability p.

(a) (0 points but you need it for later) What is $E(|X|)$? It will be a function of n, p.

(b) (0 points but you need it for later) Let $Y \subseteq V - X$ be the vertices that DO NOT have an edge to an element of X. Formally

$$Y = \{y \in V - X \colon (\forall x \in X)[(x, y) \notin E].$$

Give an upper bound on $E(|Y|)$. It will be a function of n, d, p. Note that $X \cup Y$ is a dominating set. We later pick p so that $|X \cup Y|$ is small.

(c) (0 points but you need it for later) What is $E(|X \cup Y|)$? (Hint: This is very easy by the linearity of expectation.)

(d) (30 points) Pick p to make $E(|X \cup Y|)$ smaller than n then give the function α. (Hint: Find an upper bound on $E(|X \cup Y|)$ and minimize that bound. Use that $(1 - p) \leq e^{-p}$.)
End of Proof of Sketch

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4. (35 points)

(a) (0 points but you need to do it for the rest of the problem) Write a program that will, given n, generate a random 2-coloring of $\binom{[n]}{2}$ by, for each edge, color it RED with prob $\frac{1}{2}$ and BLUE with prob $\frac{1}{2}$.

(b) (0 points but you need to do it for the rest of the problem) Write a program that will, given n and a 2-coloring of $\binom{[n]}{2}$, count how many mono K_3's there are.

(c) (0 points but you need to do it for the rest of the problem) This problem just puts the two programs together. Write a program that will, given n, generate a random 2-coloring of $\binom{[n]}{2}$ by, for each edge, color it RED with prob $\frac{1}{2}$ and BLUE with prob $\frac{1}{2}$ and then OUTPUTS the number of mono K_3's.

(d) (0 points but you need to do it for the rest of the problem) This problem mostly uses the last program. Write a program that will, given n, run the last program 100 times and gather up the number-of-triangles. Then just output the MIN and the MAX.

(e) (35 points) Write a program that outputs a table with

- n going from 4 to 30
- Min number of mono K_3's from last program.
- Max number of mono K_3's from last program.
- How much bigger is the Max from what we are guaranteed by the theorems in class. We call this *over*.

Here is the format of the table (I made up the numbers and only give the first 3 rows):

n	min	max	over
4	1	2	2
5	1	2	1
6	2	8	3

(f) (0 points) Make a conjecture about what happens for a random coloring.