BILL, RECORD LECTURE!!!!

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Lower Bounds on $W(3, c)$

Exposition by William Gasarch

VDW's Theorem

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Proof uses very hard math.

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k	2 colors	3 colors	4 colors
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- **Research Idea** Use AI/ML/SAT-Solvers to find VDW numbers.

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Want lower bounds to see how close they are to upper bounds.

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t is called **the shift**.

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Alas, we do not live in an ideal world.

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As Soren suggested, on the next slide we pick them randomly.

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Note $\frac{V\ln(V)}{|A|}$ is close to the ideal of $\frac{V}{|A|}$.

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Not so Fast We need to find 3-free sets.

3-Free Set

Exposition by William Gasarch

3-Free Set Facts

- If A is not 3-free then there exists $a, a+d, a+2d \in A$.
- If A is not 3-free then there exists $x, y, z \in A$ such that $x + z = 2y$.
- **Notation** The size of the largest 3-free set of $[V]$ is denoted $\mathrm{sz}(V)$.

$\mathrm{sz}(V) \geq V^{0.63}$

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$x = x_L \cdots x_0$
$z = z_L \cdots z_0$
$y = y_L \cdots y_0$

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Leave it to the reader to work it out.

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We denote the ith block as B_i, a number.

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$B_3 = 1$
$B_4 = 5$

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A is the set of all $B_r B_{r-1} \cdots B_1$ such that:

1. For $1 \le i \le r-2$ the leftmost bit of B_i is 0. This leads to carry-free addition.
2. $\sum_{i=1}^{r-2} B_i^2 = B_r B_{r-1}$ (The $B_r B_{r-1}$ is concatenation.)

We leave it to the reader to prove that $|A|$ is as big as we said (this is easy) and that the set is 3-free (This requires some thought.)

Back to $W(3, c)$

Recall that we prove:

Thm Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Then there is a $\frac{V \ln(V)}{|A|}$-coloring of $[V]$ with no mono 3-APs. Hence $W(3, \frac{V \ln(V)}{|A|}) \geq V$.

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Combine these two to get:

Thm Let $V \in \mathbb{N}$. Then there is a $V^{\frac{1}{\sqrt{\lg V}}}\ln(V)$-coloring of $[V]$ with no mono 3-APs. Hence

$$W(3, V^{\frac{1}{\sqrt{\lg V}}}\ln(V)) \geq V.$$