

BILL, RECORD LECTURE!!!!

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Goodstein Sequences

Exposition by William Gasarch-U of MD

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Writing a number as a sum of powers of 2.

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$$1000 = 2^{2^3+2^0} + 2^{2^3} + 2^{2^2+2^1+2^0} + 2^{2^2+2^1} + 2^{2^2+2^0} + 2^{2^1+2^0}$$

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We can even write the exponents that are not already powers of 2 as sums of powers of 2.

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This is called **Hereditary Base n Notation**

Ackermann's Function and Goodstein Seq

$$1000 = 2^{2^{2^{2^0} + 2^0} + 2^0} + 2^{2^{2^1 + 2^0}} + 2^{2^2 + 2^{2^0} + 2^0} + 2^{2^2 + 2^{2^0}} + 2^{2^2 + 2^0} + 2^{2^{2^0} + 2^0}$$

Replace all of the 2's with 3's:

$$3^{3^{3^{3^0} + 3^0} + 3^0} + 3^{3^{3^1 + 3^0}} + 3^{3^3 + 3^{3^0} + 3^0} + 3^{3^3 + 3^{3^0}} + 3^{3^3 + 3^0} + 3^{3^{3^0} + 3^0}$$

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This number just went WAY up. Now subtract 1.

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Repeat the process:

Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, \dots

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Note Does the sequence:

- ▶ Goto infinity (and if so how fast- perhaps Ack-like?)
- ▶ Eventually stabilizes (e.g., goes to 18 and then stops there)
- ▶ Cycles- goes UP then DOWN then UP then DOWN ...

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Really? Really!

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$$(95(47))_{88} \rightarrow \cdots \rightarrow (900)_y$$

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Now its a 2-digit number and use induction.

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3) **Natural Mathematical Statement** We'll take this to be a statement of interesting math content. We will soon discuss an unnatural math statement.

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Next Slide will indicate why am asking this.

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3. Harvey Friedman has done much research on this. Here is one of his theorems:

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