

# BILL, RECORD LECTURE!!!!

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# All 2-Coloring Of the Plane have a Red 2-Stick or Blue 3-stick

Exposition by William Gasarch-U of MD

# Credit Where Credit is Due

The main result in these slides is due to Szlam (1999).

# Chromatic Number of the Plane: Review

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- 4) Given a  $\text{COL}: \mathbb{R}^2 \rightarrow [2]$  a **Red**  $\ell_k$  is an  $\ell_k$  where all the points in it are **Red**. Similar for a **Blue**  $\ell_k$

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Answer on next slide

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**Case 2** There is no **Blue**  $l_3$ .

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This is well defined because of the case we are in.

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## Can Prove Result About $(l_2, l_4)$

Using that the Chromatic Number of the Plane ( $\chi$ ) is  $\leq 4$  one can easily prove the following:

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Juhasz prove the above theorem without using  $\chi \leq 4$ . In fact, he prove the theorem in 1979, 39 years before  $\chi \leq 4$  was proven.

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We will soon present statements about  $\mathbb{R}^2 \rightarrow (\ell_2, \ell_n)$

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Csizmadia-Togh 1994	$\exists X \subseteq \mathbb{R}^2,  X  = 8$ $\mathbb{R}^2 \not\rightarrow (l_2, X)$	NONE

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Conlon-Fox, 2018	$(\forall n \geq 2)$ $\mathbb{R}^n \not\rightarrow (l_2, l_{10^{10}})$	$\mathbb{R}^2 \not\rightarrow (l_2, l_{10^{50}})$

$\mathbb{R}^2 \rightarrow (\ell_2, \ell_n)$ ?

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**Open** Narrow the gap between  $\mathbb{R}^2 \rightarrow (\ell_2, \ell_5)$  and  $\mathbb{R}^2 \not\rightarrow (\ell_2, \ell_{10^{50}})$ .

$$\mathbb{R}^2 \rightarrow (l_3, l_n)?$$

All of the negative results hold for all  $n \geq 2$ . Our interest is in the  $n = 2$  case.

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Currier-Moore-Yip, 2024	$\mathbb{R}^2 \rightarrow (l_3, l_3)$
<b>Negative Results</b>	

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The results of Fuhrer-Togh and Currie-Moore-Yip are messy. For some reasonable values of  $n$  find nice proofs that  $\mathbb{R}^2 \not\rightarrow (\ell_3, \ell_n)$ .

# One More Result

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The title of the paper is **Euclidean Ramsey Theorems I (1973)** and was (obviously) an early paper on Euclidean Ramsey Theory.

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**Open** In some cases get nicer proofs of weaker results.

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