Cheat Sheet for $\mathsf{R}^2 \not\to (\ell_3, \ell_{10^{50}})$

1 Notation

Notation 1.1. Let $a, b \ge 2$. $\mathsf{R}^2 \to (\ell_n, \ell_m)$ means
for all COL: $\mathsf{R}^2 \to [2]$ either
(1) there exists a Red ℓ_n or (2) there exists a Blue ℓ_m.

2 Goal

We need COL$'$: $\mathsf{R} \to [2]$ such that
1) No RED solution to

$$y_1 + y_3 = 2y_2 + 2.$$

2) No BLUE solution to
forall $2 \le i \le m-1$,

$$y_{i-1} + y_{i+1} = 2y_i + 2.$$

We have not determined m yet. We will later.
However we will require that $m = q^3$ where q is prime.

3 Image-Solutions

Let $p_1(z_1, z_2)$, …, $p_m(z_1.z_2) \in \mathsf{Z}[z_1, z_2]$. We denote this set of polynomials by F.

1) An *image-solution of F* is just $(a, d; y_1, \ldots, y_m)$ where $(\forall 1 \le i \le m)[y_i = p_i(a, d)]$.

2) Assume the reals are 2-colored. Then a *mono image-solution of F* is an image-solution $(a, d; y_1, \ldots, y_m)$ where $y_1, \ldots, y_m$ are all the same color. Note that a, d need not be that color.

3) A *Red image-solution of F* and a *Blue image-solution of F* obvious.

Why We Care There exists F such that for all 2-coloring of R
no blue image-solution to F IFF no blue solution to $(\forall 2 \le i \le m-1)[_{i-1} + y_{i+1} = 2y_i + 2]$.

4 COL, COL$'$, COL$''$

Assume COL$''\colon \mathsf{Z}_q \to [2]$:

1) COL$''$ has no RED solution (in Z_q) to $y_1 + y_3 = 2y_2 + 2$.
2) COL$''$ has no BLUE solution (in Z_q) to $(\forall 2 \le i \le m-1)[y_{i-1} + y_{i+1} = 2y_i + 2]$.
(Actually will get no blue image-solution to F.)

Let COL$'\colon \mathsf{Z} \to [2]$ be COL$''(\lfloor y \rfloor \pmod q)$. Can show
1) COL$'$ has no RED solution (in Z) to $y_1 + y_3 = 2y_2 + 2$.
2) Has no BLUE solution (in Z) to $\forall 2 \le i \le m-1)[y_{i-1} + y_{i+1} = 2y_i + 2]$.

Let COL$\colon \mathsf{R}^2 \to [2]$ be COL$(\vec{a})$ = COL$'(d(0,\vec{a}))$. Did show
1) COL has no RED ℓ_3 (in R^2).
2) COL has no BLUE ℓ_m (in R^2).

5 The Coloring

We will pick COL$\colon \mathsf{Z}_q \to [2]$ randomly.

We will not color each element RED or BLUE with equal probability.

We want RED to be far rarer than BLUE.

We pick
Prob of a RED to be $p = q^{-3/4}$. Prob of BLUE to be $1-p$

6 Intervals

Lemma 6.1. *Let $p(x) = x^2 + \alpha x + \beta$ where $\alpha, \beta \in \mathsf{R}$. Let q be a prime. Let $f(x) = p(x) \pmod q$. Let $m \ge q^3$.*

$$\text{Let } X = \{f(1), f(2), \ldots, f(m)\}.$$

Then
X hits at least $q/6$ of the intervals $[0,1)$, $[1,2)$, …, $[q-1,q)$.

7 Sign Patterns

Let $p_1, \ldots, p_M \in \mathsf{R}[x,y]$.
Let $X = (p_1, \ldots, p_M)$.
$\eta \in \{-,0,+\}^M$ is a *sign pattern for X* if
there exists $a_1, a_2 \in \mathsf{R}$ such that for all $1 \le i \le M$

$$\mathrm{sign}(p_i(a_1,a_2)) = \eta(i).$$

Obvious bound on number of sign patterns: 3^M

Lemma 7.1. *Let $M \in \mathsf{N}$. Let $p_1, \ldots, p_M \in \mathsf{Z}[x,y]$. The number of sign patterns is $\le 25M^2$.*