

There Is a 2-Coloring Of the Plane Without a mono Red 3-Stick or a mono Blue Big-Stick

Exposition by William Gasarch-U of MD

Credit Where Credit is Due

The main result in these slides is due to Conlon and Wu (2022).

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Main Theorem

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Open Find an easier proof of $\mathbb{R}^2$ case.

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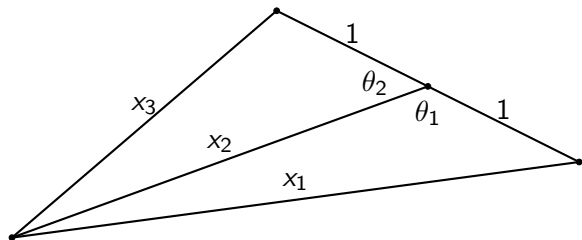
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And we know

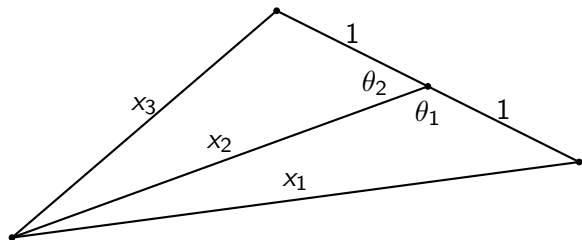
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Algebra and Trigonometry

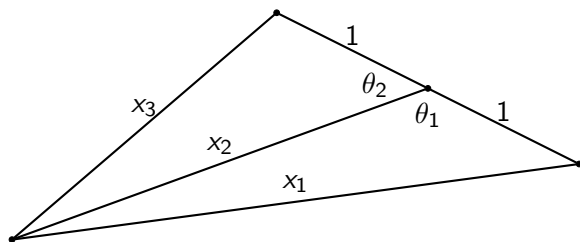


Algebra and Trigonometry



Bottom Triangle:

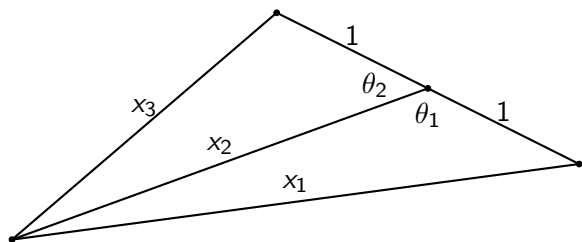
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Law of cosines: $x_1^2 = x_2^2 + 1 - 2x_2 \cos(\theta_1)$.

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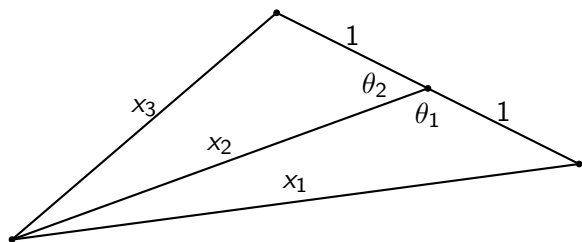


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Add to get

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

First Plan On How to Avoid R_{l_3}

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Can get rid of squares.

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We will also have a condition on COL' that will make $\text{COL}(\vec{a}) = \text{COL}'(d(\vec{0}, \vec{a}))$ not have any $\mathbf{B} \ell_m$

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Real Plan On How to Avoid B_{lm}

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Real Plan

Real Plan On How to Avoid $\mathbf{B}$ ℓ_m

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Real Plan On How to Avoid $\mathbf{B}$ ℓ_m

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An Example of A Coloring with $q = 5$

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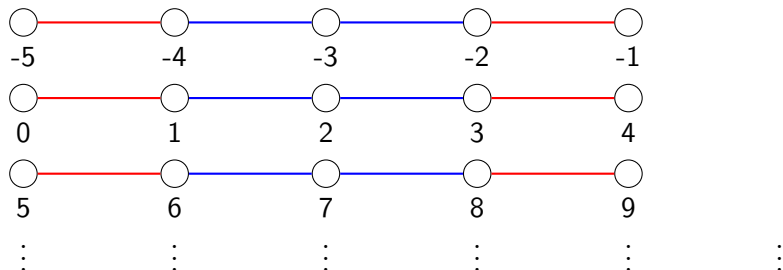
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The next slide recaps where we are and says why COL'' helps us.

COL'' and COL' and COL

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Lemmas and a Theorem of Independent Interest

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What does $p(x) = x^2 + \pi x + e \pmod{13}$ mean?

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So it makes sense to consider $p(x) \pmod{q}$ where $p(x) \in \mathbb{R}[x]$.

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X hits at least $q/6$ of the intervals $[0, 1), [1, 2), \dots, [q-1, q)$.

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q is a prime so squaring is 2-to-1. Hence $|\text{SQ}_q| = (q+1)/2$.

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Recap There is a $k \not\equiv 0 \pmod{q}$ such that $|k\alpha \pmod{q}| \leq \frac{1}{q}$.

Plan

1) Show $x^2 + \beta \pmod{q}$ hits $\geq (q+1)/2$ intervals.

2) Show that adding αx has a small effect since

$$|k\alpha \pmod{q}| \leq \frac{1}{q}.$$

We consider several sets and see how many intervals they hit.

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Since every element in SQ_q is an integer, hits $(q+1)/2$ intervals.

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Since $k \not\equiv 0 \pmod{q}$, $\{k, 2k, \dots, qk\} = \{1, 2, \dots, q\}$. Hence $X = Y$.

Why $m = q^3$?

We have shown that

$$\{f_1(k), f_1(2k), \dots, f_1(qk)\}.$$

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We will do this on the next slide.

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$\{f(k), f(2k), \dots, f(qk)\}$:

$f(k) = f_1(k) + k\alpha$. **Key** Recall $|k\alpha \pmod{q}| \leq \frac{1}{q} \leq 1$.

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Recap The set $Y = \{f_1(k), \dots, f_1(qk)\}$ hits $(q+1)/2$ intervals of length 1.

$Z = \{f(k), \dots, f(qk)\}$ can be viewed as taking every element in Y and adding or subtracting ≤ 1 to it. It is easy to show that Z hits $\geq q/6$ intervals.

Case 2: $k \equiv 0 \pmod{q}$

OMITTED FOR NOW.

Another Lemma Of Independent Interest

The Sign Function and Other Notation

Def if $a \in \mathbb{R}$ then

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Notation If $\eta \in \{-1, 0, 1\}^*$ then $\eta(i)$ is the i th character in η .

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(x, y)	$(p_1(x, y), p_2(x, y), p_3(x, y))$	sign pattern
$(0, 0)$	$(-3, -7, 0)$	$(-, -, 0)$
$(10, 0)$	$(7, -27, 40)$	$(+, -, +)$
$(0, 10)$	$(17, 23, -10)$	$(+, +, -)$
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I doubt it has anywhere near 27.

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Lemma About Sign Patterns

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Proof Omitted thought it is not hard.

Lemma also a corollary of a theorem by
Olenik-Petrovsky-Thom-Milnor.

Image-Solution

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- 3) A **Red image-solution of F** and a **Blue image-solution of F** obvious.

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I am skipping some stuff of interest to get to some stuff that is of more interest.

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prob there is a **blue image-solution** to $p_1, \dots, p_m$ is
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Number of ways $p_1(a, d), \dots, p_m(a, d)$ are in the ints is $\leq 2500m^6$.

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Hence there exists a 2-coloring with no **R** ℓ_3 or **B** ℓ_m for large enough m .