BILL, RECORD LECTURE!!!!

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Coloring $\binom{Z}{2}$

Exposition by William Gasarch-U of MD

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$$\{1 < 2 < 3 < 4 < \cdots < \omega < \omega + 1 < \omega + 2 < \cdots\}$$

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You are on your honor to not go to the next slide which has the answer.

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The color of a pos-pos is R. The color of neg-neg is B.
Different colors. Not homog.

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$\forall\ \mathrm{COL}\binom{\mathbb{Z}}{2} \to [c]$ $\exists$ **infinite c-homog set $H \equiv \mathbb{Z}$.**

We want examples where $c < d$.

Questions For You

We want to have a $\mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [d]$ and get an inf c-homog $H \equiv Z$ with $c < d$.
Work in groups to see if the following are TRUE or FALSE.

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There is no 3-homog $H \equiv \mathbb{Z}$. Left to the reader.

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$$\forall\, \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \to [1,000,000]\ \exists\, H \equiv \mathbb{Z},\ H \textit{ is 4-homog.}$$

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4) We need a thm about bipartite graphs.

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Def $K_{\mathbb{N},\mathbb{N}}$ is the bipartite graph $(\mathbb{N}, \mathbb{N}, \mathbb{N} \times \mathbb{N})$.

Note A coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ is a coloring of $\mathbb{N} \times \mathbb{N}$.

Infinite Ramsey Theory for $K_{\mathbb{N},\mathbb{N}}$

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We use $\mathrm{EVEN}^+ \times \mathrm{ODD}^+$ instead of $\mathbb{N} \times \mathbb{N}$.

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There is no 1-homog $H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$. Left to the reader.

$c \le 2$

$\forall\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [\mathbf{1,000,000}]\ \exists\ \mathbf{H_1} \times \mathbf{H_2}$**, 2-homog.**

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$\forall$ COL: $\mathbb{N} \times \mathbb{N} \rightarrow [\mathbf{1,000,000}]$ $\exists$ $\mathbf{H_1 \times H_2}$, **2-homog.**

We do an example. The formal construction is left to the reader.

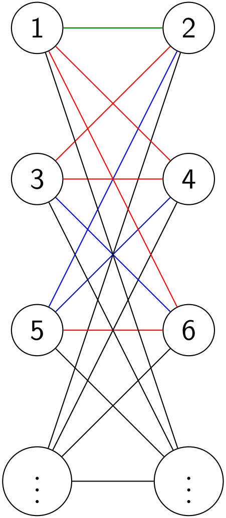
$c \le 2$

$\forall\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]\ \exists$ $\mathbf{H_1} \times \mathbf{H_2}$**, 2-homog.**

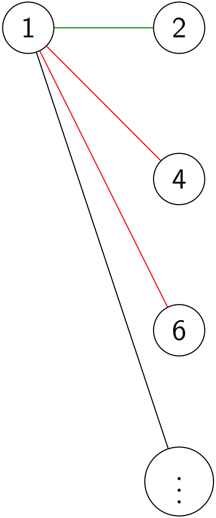
We do an example. The formal construction is left to the reader.

Initially we have $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]$.

Example of finite coloring of $\mathbb{N} \times \mathbb{N}$



Focus on Vertex 1 On The Left



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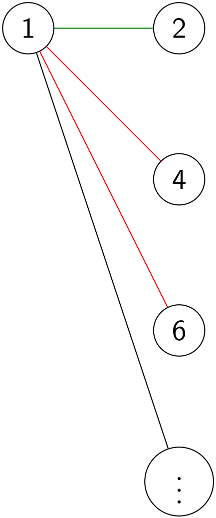

Let c be least color such that $\exists^{\infty} x, \mathrm{COL}(1, x) = c$. We assume R.

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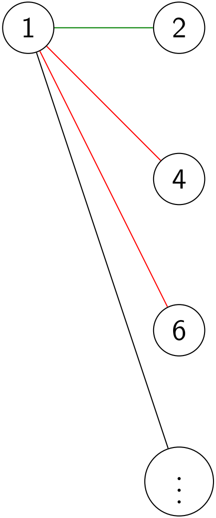

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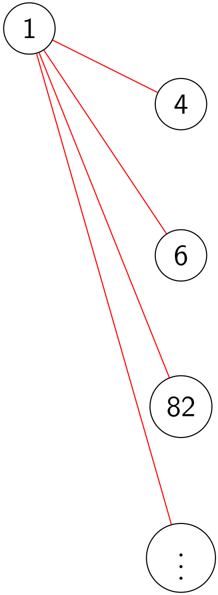
Kill All Those On The Right Who Disagree.

Focus on Vertex 1 On The Left After The Massacre



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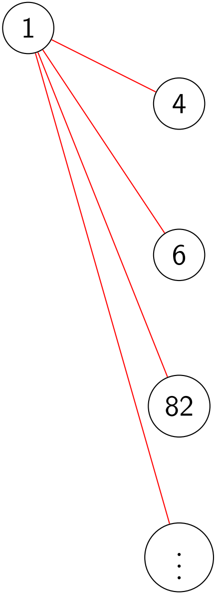
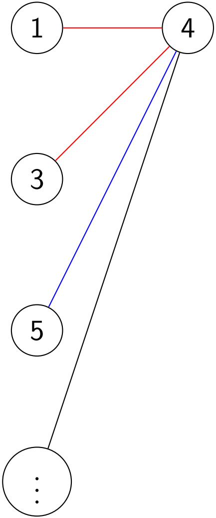
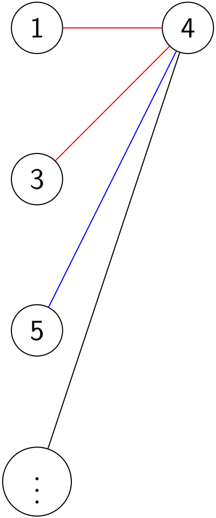


1 is "**immortal**". We focus on 4.

Focusing on 4 On The Right

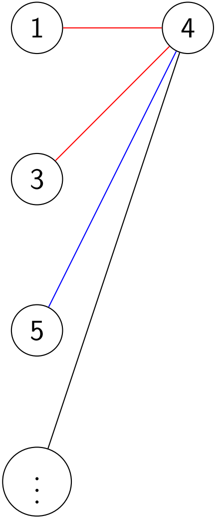


Focusing on 4 On The Right



Let c be least color such that $\exists^{\infty} x, \mathrm{COL}(x, 4) = c$. We assume B.

Focusing on 4 On The Right



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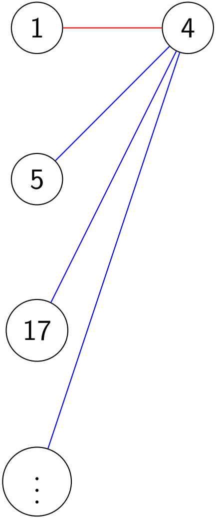

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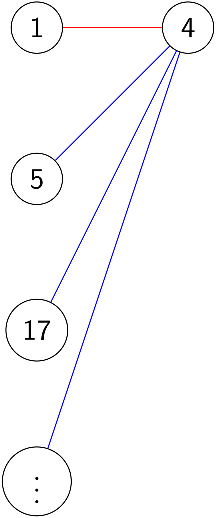

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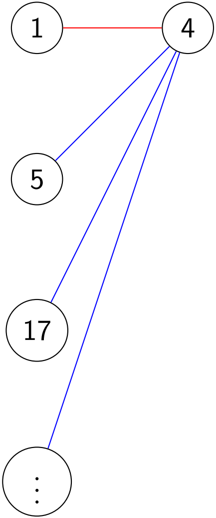

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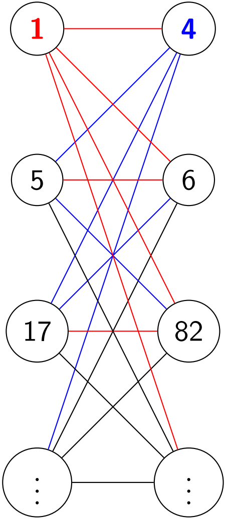
We Have Processed 1 and 4



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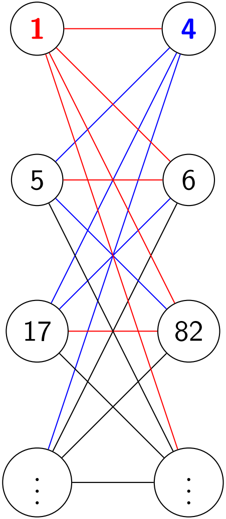

1 is colored *R*. 4 is colored *B*. 1,4 "**immortal**"

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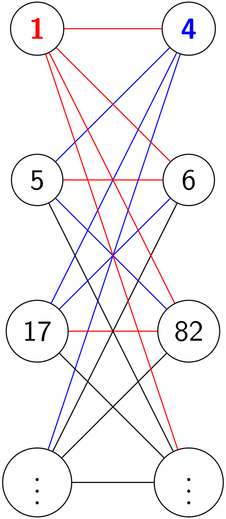

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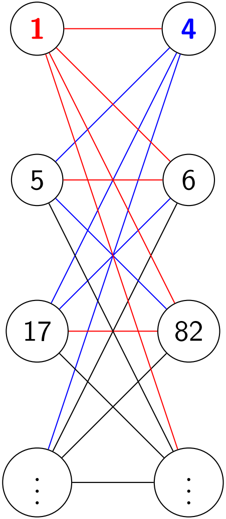
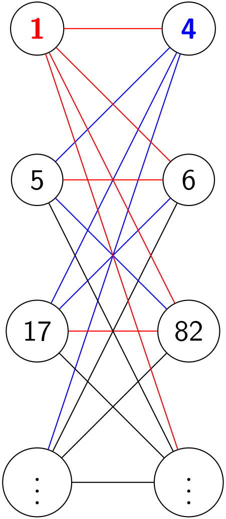


1 is colored R. 4 is colored B. 1,4 "**immortal**"

4 is B even though one of the edges out of it is R.

Key If $x > 4$ then $\mathrm{COL}(x, 4) = B$.

We Have . . ., Now What?



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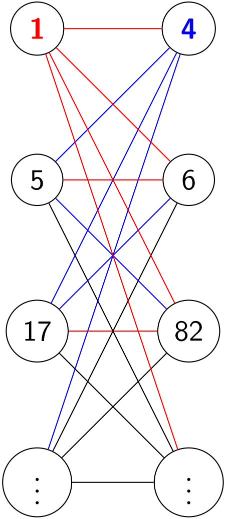

Process 5, Color 5, Process 6, Color 6. Both “**immortal**”

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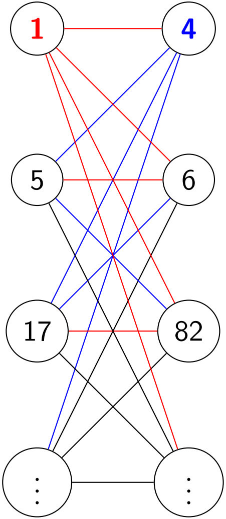

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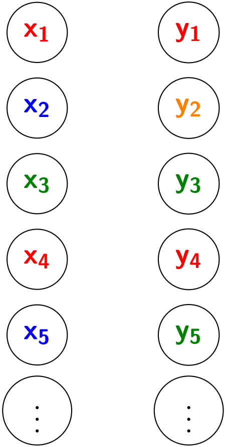
Repeat Process.

So You Thought You Were Immortal. HA!



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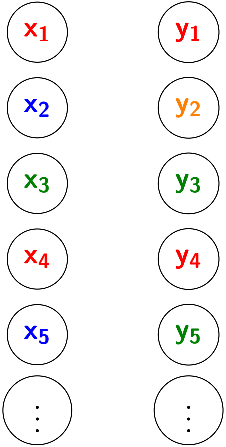

Finite number of colors (though could be large)

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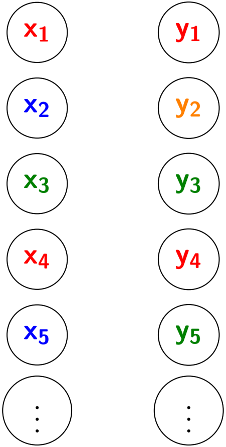

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$(\exists c)(\exists^\infty x)[\mathrm{COL}'(x) = c]$. Assume R.

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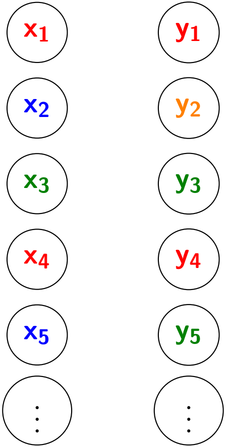

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$(\exists d)(\exists^{\infty} x)[\mathrm{COL}'(y) = d]$. Assume G.

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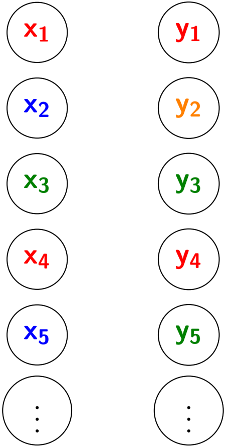

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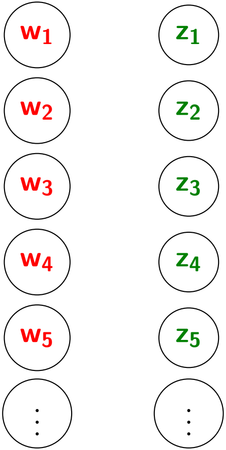
We Are Done!



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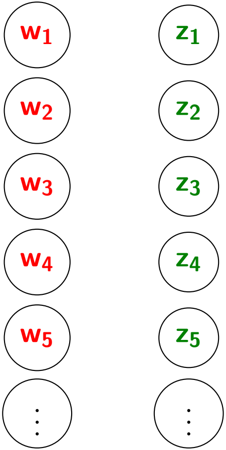
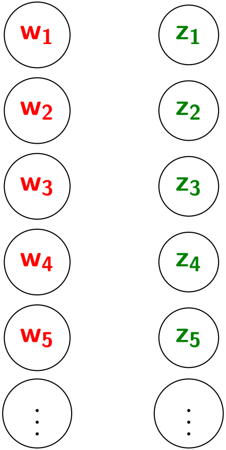


$$H_1 = \{w_1, w_2, \ldots\}$$

We Are Done!



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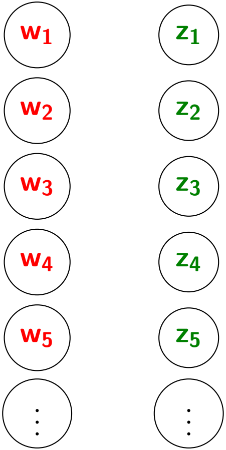

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Began with finite $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]$.

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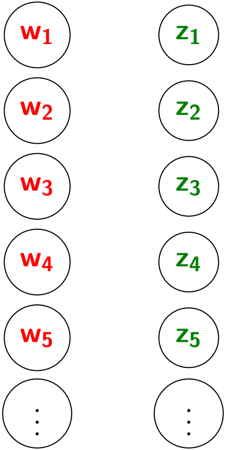

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$\mathrm{COL}\colon H_1 \times H_2$ uses only two colors.

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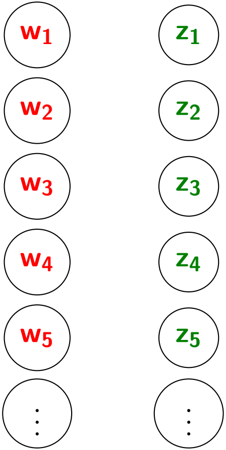

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Restate theorem we just proved on next slide.

Recap

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Thm $\exists \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [2]$ such that there is no 1-homog (H_1, H_2).
Thm $\forall \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]$ $\exists$ 2-homog (H_1, H_2).

Back to $\mathbb{Z}$

Theorem for $\mathbb{Z}$

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Thm 2 we prove on the next slide.

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At most 4 colors. DONE!

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$$\omega^2 = \omega + \omega + \omega + \cdots$$

Theorem for $\omega + \omega$

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Proof that you need 4 colors similar to that for $\binom{\mathbb{Z}}{2}$.

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Lots of linear orders have been looked at.

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Some of the other results might be on a HW.