BILL, RECORD LECTURE!!!!

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"Applications" or Ramsey Theory

Exposition by William Gasarch-U of MD

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3) Applications to Ramsey Theory.

Applications Outside of Combinatorics

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This is **not** one of those times.

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CAVEAT Sora is not impressed.

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CON They only care about it as an example of thm that requires a certain proof strengths to prove.

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CAVEAT The p is enormous as a function of p. I suspect there is a harder NT proof that gives a much better bound.

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PRO There is a k-people version that also has matching Ramsey-type bounds.

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Most of the seven then go on to make other points.

Applications

To Combinatorics

But Outside of Ramsey Theory

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BIG PRO It brought Esther Klein and George Szekeres together!

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Question Are you down with that? (We need a rap song about wqo.)

Applications To Ramsey Theory

Recall: Hungarian Math Comp Problem

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I used this to motivate the Can Ramsey Theorem.

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Orban, the president of Hungary, has demanded the math community there stop using the term **rainbow** since it is associated with Gay Rights.

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CAVEAT I prefer the name **Can Ramsey Thm**.

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PRO Nice application of **discrete** Ramsey to **continuous** Ramsey.