

BILL, RECORD LECTURE!!!!

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Euclidean Ramsey Theory Chromatic Number of the Plane

Exposition by William Gasarch

May 1, 2026

Ramsey Theory VS Euclidean Ramsey Theory

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In Euclidean Ramsey Theory we will be coloring the Plane or \mathbb{R}^d .

B) We do not care about the geometric size. For example, the Square can be any size.

In Euclidean Ramsey Theory we will be seek an object of a certain size, for example the unit square.

For All 2-Colorings of the Plane...

Thm $\forall \text{COL}: \mathbb{R}^2 \rightarrow [2] \exists 2 \text{ points, same color, 1 inch apart.}$

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Proof on the next page.

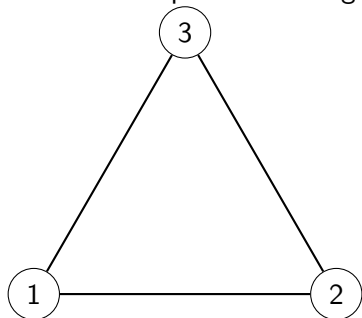
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Look at an equilateral triangle in the plane

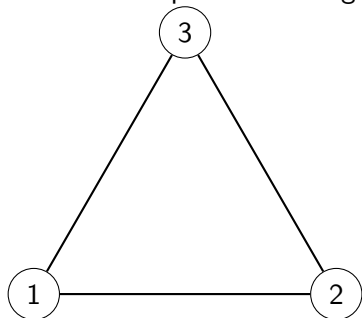
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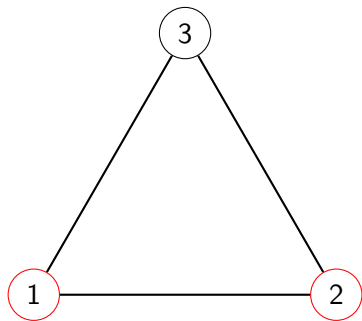
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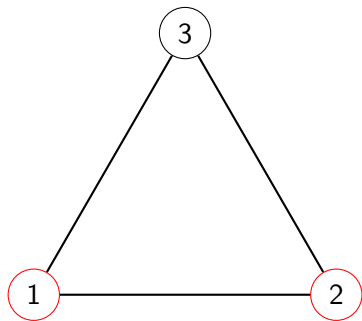
3 vertices and 2 colors. So 2 of the vertices are the same color.

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Vertices 1 and 2 are an inch apart.

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We investigate what χ can be.

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Answer on next slide

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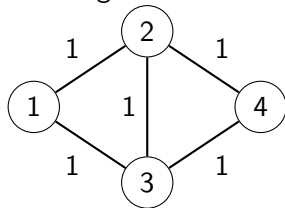
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Glue together two unit equilateral triangles:

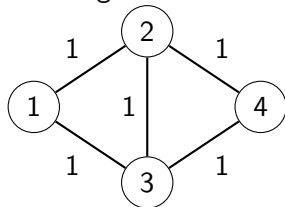
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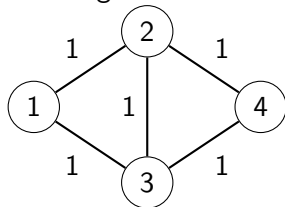
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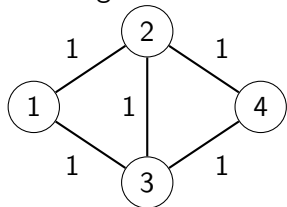
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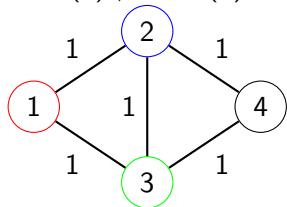
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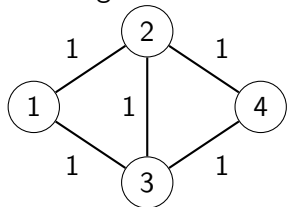


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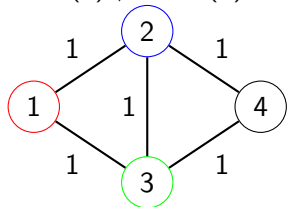


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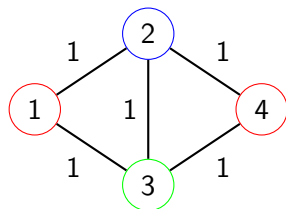
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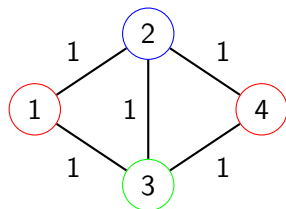
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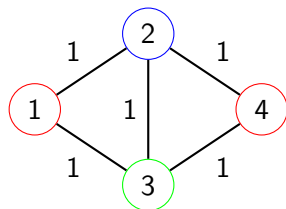


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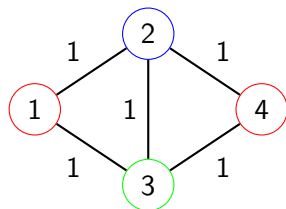
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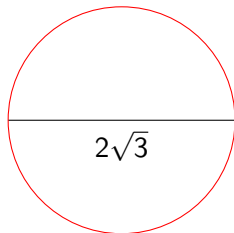
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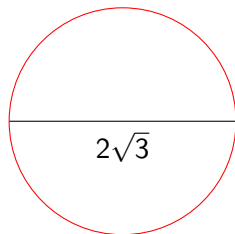
Upshot 2 If $\text{COL}(p) = \mathbf{R}$ then circle of radius $\sqrt{3}$ around p is \mathbf{R} .

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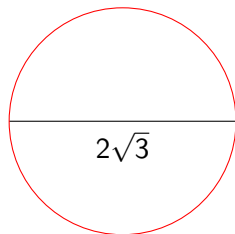


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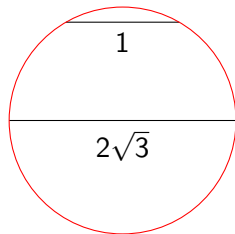


Look at a chord of the circle of length 1.

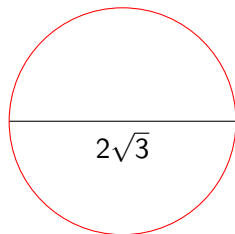
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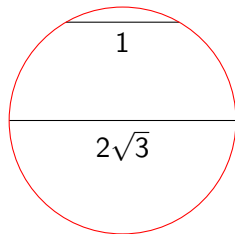
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Endpoints of chord are **R** and an inch apart.

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It is a 7-vertex graph drawn in the plane with all sides of length 1.

It is an easy exercise to show that this graph is not 3-colorable.

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So we know that $\chi \geq 5$.

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Here is the 7-coloring:

[https://thatsmaths.com/2022/03/24/
the-chromatic-number-of-the-plane/](https://thatsmaths.com/2022/03/24/the-chromatic-number-of-the-plane/)