Constructive Ramsey Lower Bounds AND Chromatic Number of $\mathbb{R}^n$!

Exposition by William Gasarch-U of MD

Credit Where Credit is Due

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3) A great set of slides by Kelin Zhu.
(An undergraduate working with me.)

Easy But Weak Lower Bounds on $R(k)$.

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$R(k) > k^{\log_4 17} = k^{2.043}$.

We Will Prove Two Combinatorial Lemmas

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If I have $(\forall i, j)$ it is understood that $i \neq j$.

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$(\forall i, j)[|w_i \cap w_j| \equiv 0]$.
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$w_i \cdot w_i \equiv |w_i| \equiv 1$.
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Rest of proof on next slide. (So write down theorem.)

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Since $w_i \cdot w_i \equiv 1$:
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So $(\forall i)[\alpha_i \equiv 0]$ so $w_1, \ldots, w_m$ are linearly independent.

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Case 4 $1 \le \lambda < v$. Next slide.

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Hence $\alpha = 0$, so $(\forall i)[\alpha_i = \alpha = 0]$.

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Note The constructive lower bound used **both** Oddtown and Fisher.

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The above results can be summarized as $5 \le \chi(G) \le 7$.

Chromatic Number of $\mathbb{R}^2$

Easy Thm For all 2-colorings of $\mathbb{R}^2$ there exists 2 points, an inch apart, that are the same color.
Slightly Harder Thm For all 3-colorings of $\mathbb{R}^2$ there exists 2 points, an inch apart, that are the same color.
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What is $\chi(\mathbb{R}^n)$? The next slide gives some information.

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4) The best known results asymptotically are

$$(1.239 + o(1))^n \le \chi(\mathbb{R}^n) \le (3 + o(1))^n.$$

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Notation We call that last theorem **Oddtown'**.

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We will use $c < \frac{(n-1)(n-2)}{6}$.

Lower Bound on $\chi(\mathbb{R}^n)$

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Hence $\chi(\mathbb{R}^n) \geq \frac{(n-1)(n-2)}{6} = \Omega(n^2)$.

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Or maybe $p = 2$ is a special case.

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Key Set of multilinear polys of degree $|L|$ is a vector space of dim $\sum_{a=0}^{|L|} \binom{n}{a}$. So need to show that the f_i's are linearly independent.

Linearly Independent: What We Need Ahead of Time

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We use all of this on the next slide.

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We Use Gen of ODD and FISH to get better Lower Bound on $R(k)$.

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We Use Gen of ODD to get better Lower Bound on $\chi(\mathbb{R}^n)$

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Notation We call the above Theorem Gen-Oddtown'

Lower bound on $\chi(\mathbb{R}^n)$

Thm Let $c < \binom{n}{2p-1} / \sum_{a=0}^{p-1} \binom{n}{a}$. $\forall \mathrm{COL}\colon \mathbb{R}^n \to [c]$
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Note Only used Gen. Oddtown.

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Better is known. See next slide.

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How do they prove things with prime powers? My first thought was that they use the finite field on p^b elements, but then they can't use mods.