BILL, RECORD LECTURE!!!!

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Goodstein Sequences

Exposition by William Gasarch-U of MD

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This is called **Hereditary Base n Notation**

Ackermann's Function and Goodstein Seq

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Replace all of the 2's with 3's:

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Replace 3 by 4, and subtract 1, Replace 4 by 5, and subtract 1, $\cdots$.

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Vote Does the sequence:

- Goto infinity (and if so how fast- perhaps Ack-like?)
- Eventually stabilizes (e.g., goes to 18 and then stops there)
- Cycles- goes UP then DOWN then UP then DOWN ....

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Really? Really!

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Then use induction on the length of the number.

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3) **Natural Mathemtical Statement** We'll take this to be a statement of intereting math content. We will soon disuss an unnatural math statement.

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Next Slide will indicate why am asking this.

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3. Harvey Friedman has done much research on this. Here is one of his theorems:
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