The Hales-Jewett Theorem

Exposition by William Gasarch-U of MD

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We will take $\Sigma = \{1, \ldots, t\}$ for the proof but we never use the numeric properties of $1, \ldots, t$.

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We have $(\forall c)[HJ(2,c)]$. We denote this $HJ(2,\omega)$.

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We prove a lemma from which the theorem will follow easily.

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3) Hence we have $HJ(t-1, c) \rightarrow HJ(t, c)$.

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3) Hence we have $HJ(t-1, c) \rightarrow HJ(t, c)$.

4) Since we know $HJ(2, \omega)$ we know $(\forall t)(\forall c)[HJ(t, c)]$.

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