BILL, RECORD LECTURE!!!!

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Infinite Ramsey Theorem For 3-Hypergraph

Exposition by William Gasarch

March 6, 2026

Notation

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There may be other infinite homog sets.

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We do an example of the first few steps of the construction.

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What to make of this? Discuss.

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Kill all those who disagree!

Construction of $x_1, H_1, c_1, x_2, H_2, c_2$

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Next Slide is General Case.

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Construction of $x_{s+1}, H_{s+1}, c_{s+1}$

Assume we have x_s, H_s, c_s.

Let x_{s+1} be the least element of H_s.

$\mathrm{COL}'\colon \binom{H_s-\{x_1,\ldots,x_{s+1}\}}{2} \rightarrow [2]$ is defined by

$\mathrm{COL}'(y,z) = \mathrm{COL}'(x_{s+1},y,z)$

Let H_{s+1} be an infinite homog set (rel to COL').

Let c_{s+1} be the color of H_{s+1}.

The Coloring of the Nodes

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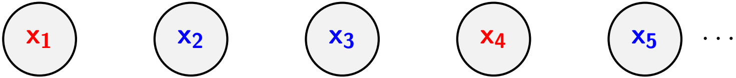

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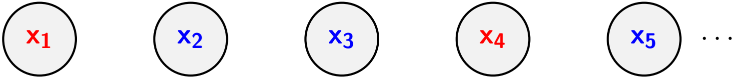

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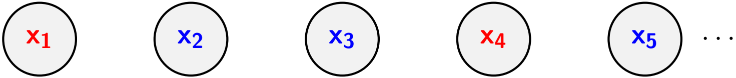

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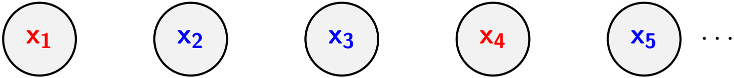

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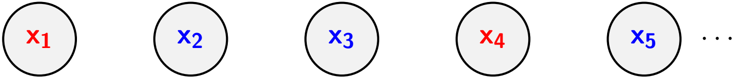

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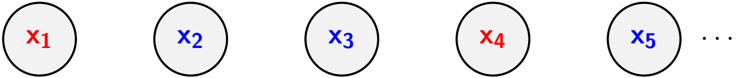

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What do you think our next step is?

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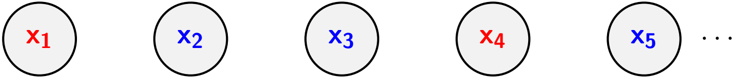

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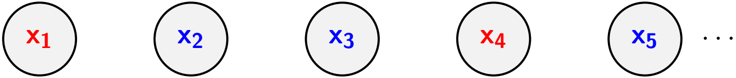

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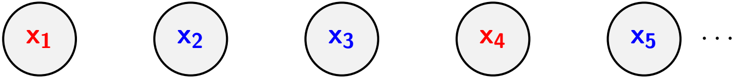

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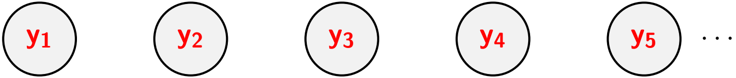

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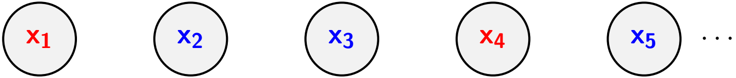

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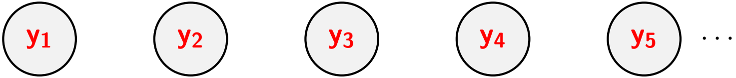

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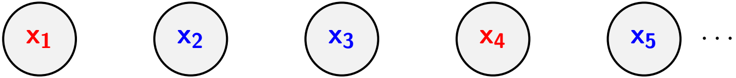

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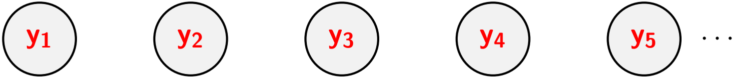

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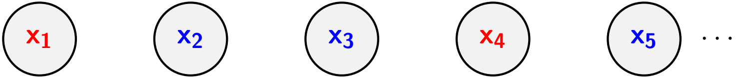

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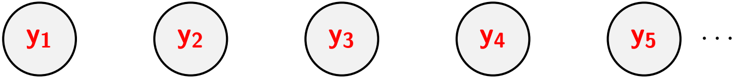

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DONE!

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$a \ge 4$: Might be a HW. Should be easy for you now.