

BILL, RECORD LECTURE!!!!

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Infinite Ramsey Theorem For Graphs

Exposition by **William Gasarch**

January 27, 2026

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Following his advice we will spend the first part of the course
discussion

Ramsey Theory on Infinite Sets

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7. The set of all unordered triples of \mathbb{N} is $\binom{\mathbb{N}}{3}$. (That will be a 3-hypergraph which we will study with later.)

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Let $\text{COL}: \binom{A}{2} \rightarrow [2]$. A set $H \subseteq A$ is **homogenous** if COL restricted to $\binom{H}{2}$ is constant. (From now on **homog.**)

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I will ask for the proof on HW 01.

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My apologies to the math majors who are not used to seeing examples.

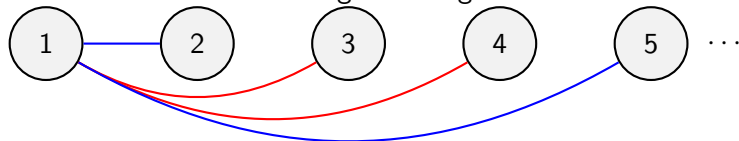
Examples of The First Few Steps of The Construction

First Step of Our Construction

Look at 1 and all of the edges coming out of it:

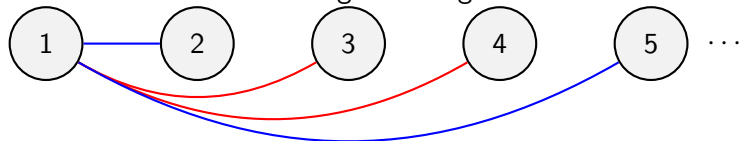
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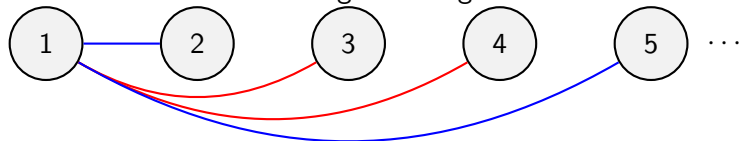
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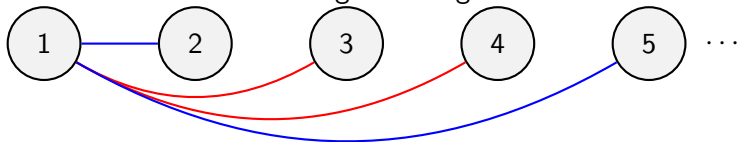
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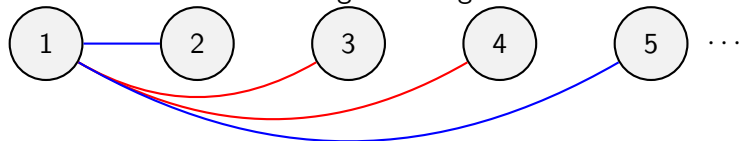
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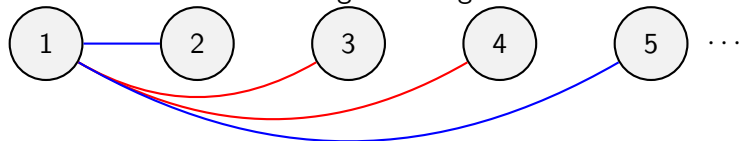
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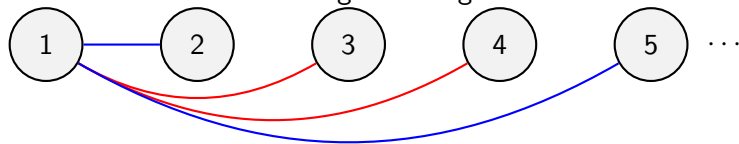
We have a picture of this on the next slide.

Node 1 Has the Reds

Look at 1 and all of the edges coming out of it:

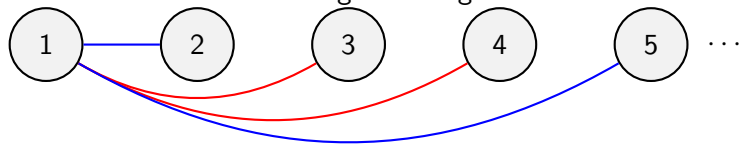
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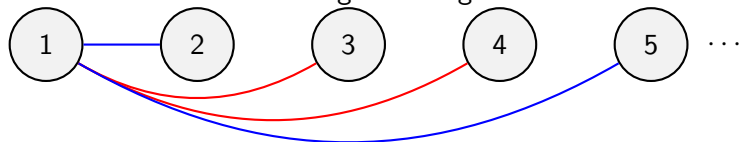
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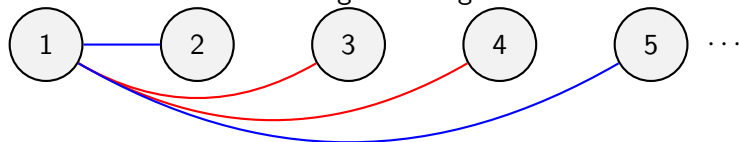
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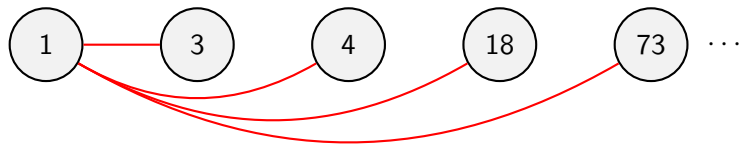
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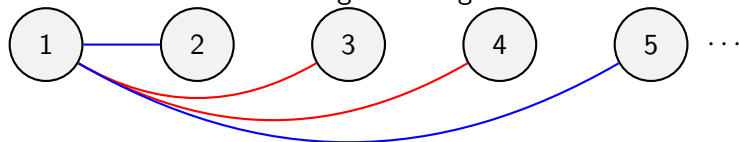


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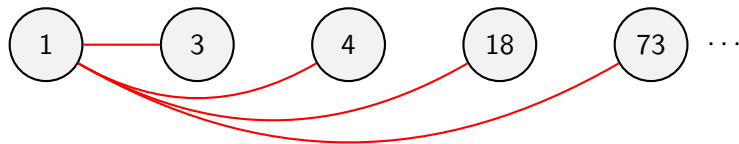


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We Omit 1 from future pictures but its **Still Alive and Well.**

<https://www.youtube.com/watch?v=8--jVqaU-G8>.

The Next Step

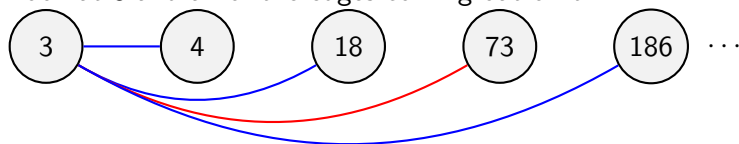
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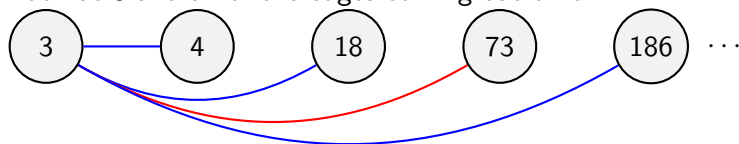
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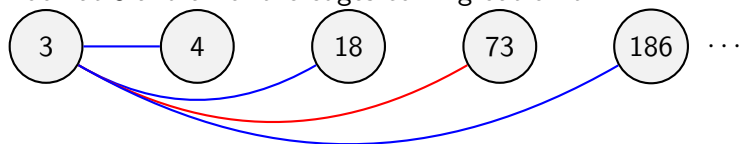


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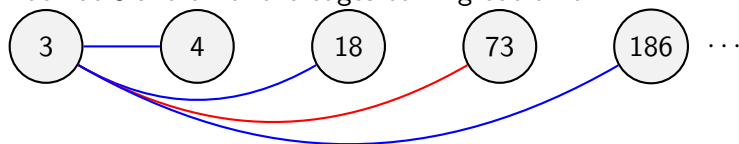
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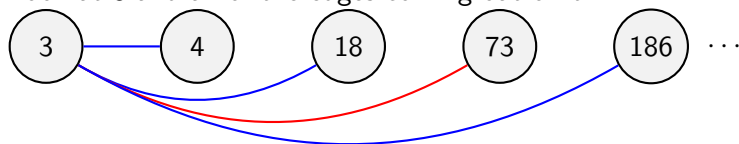
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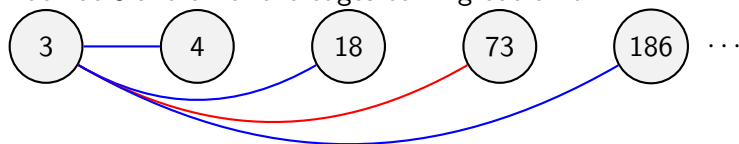
If $\text{COL}(3, y) \neq \mathbf{B}$ we say that y **disagrees**.

Kill all those who disagree! (except 1 which is immortal-for now).

The Next Step

There is a **R** edge from 1 to 3, 4, 18, 73, 186, ...; however, this puts no constraint on the colorings between those nodes.

Look at 3 and all of the edges coming out of it.



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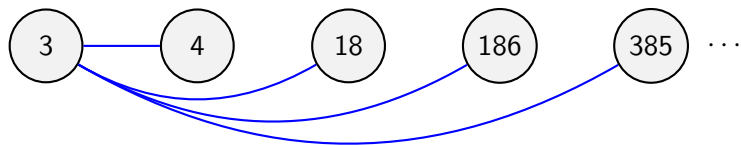
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We have a picture of this on the next slide.

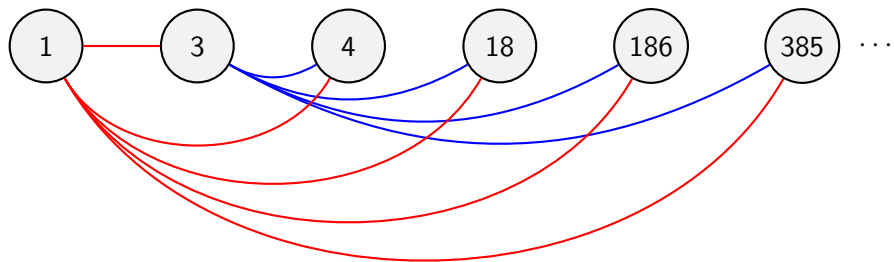
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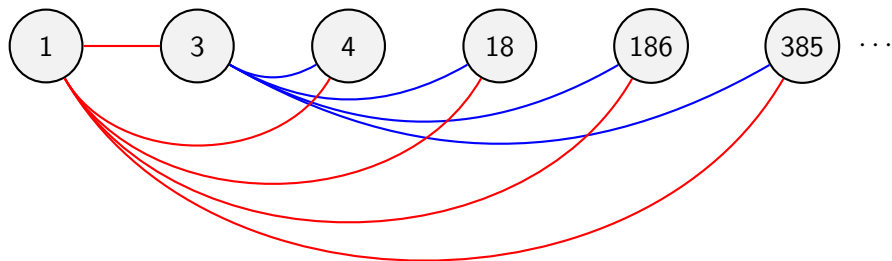


Node 1 and Node 3

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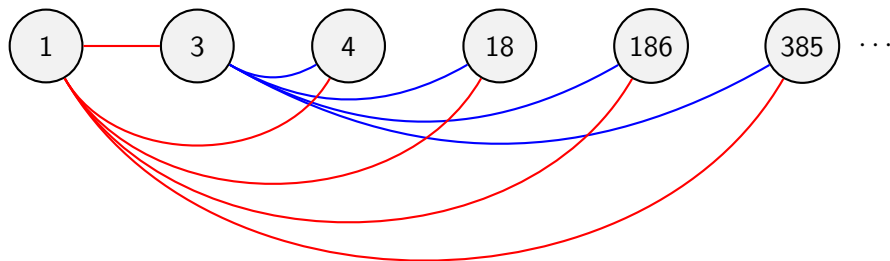


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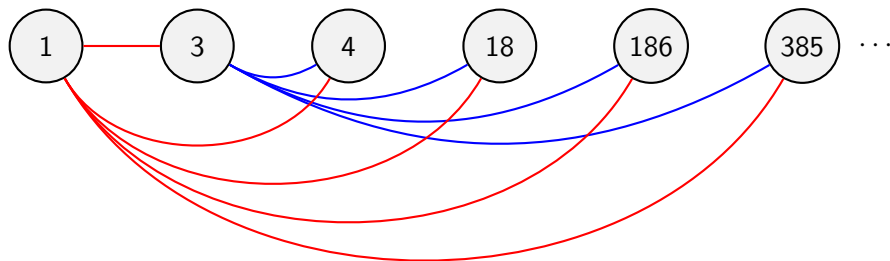
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We formalize the real construction on the next slides.

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We said earlier

Either $\exists^\infty \mathbf{R}$ or $\exists^\infty \mathbf{B}$ coming out of 1

(Or both, in which case use \mathbf{R} for what follows.)

When we formalize this, we will **color** node 1 with that color.

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Still have an Infinite Number of Nodes In Play.

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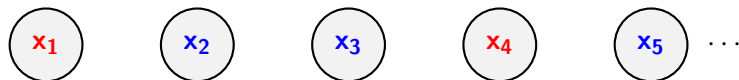
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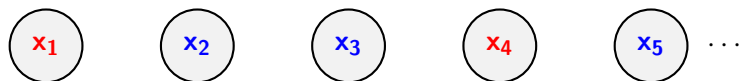
The Coloring of the Nodes

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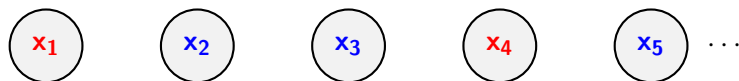
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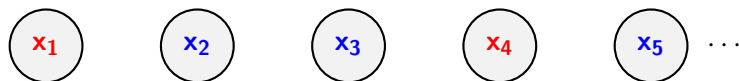


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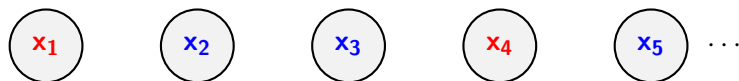
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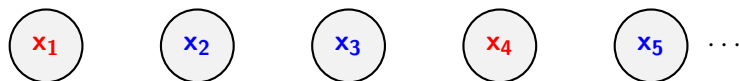
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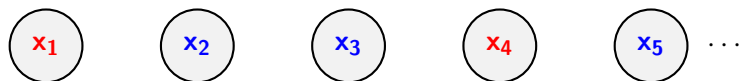
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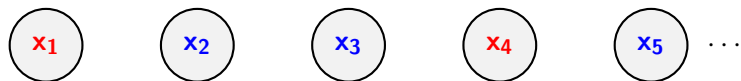
All of the edges from x_5 to the right are **B**.

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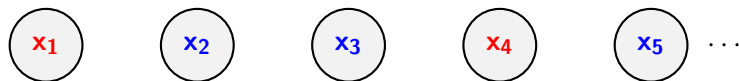
What do you think our next step is?

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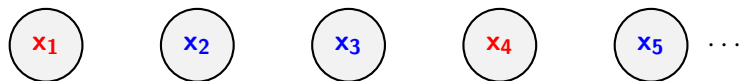


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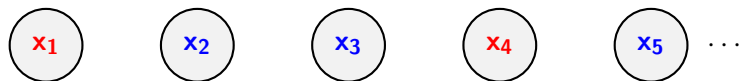
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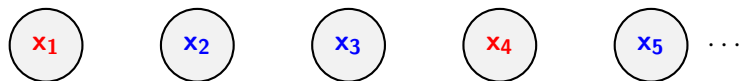


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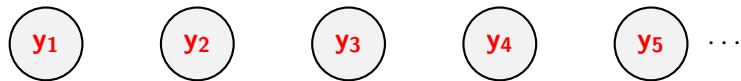
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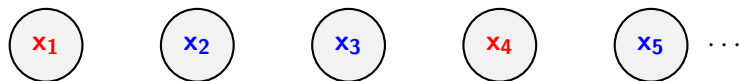
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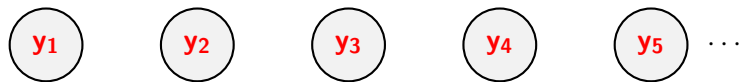
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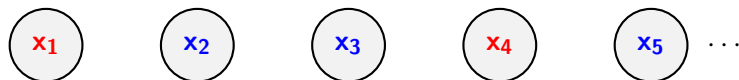
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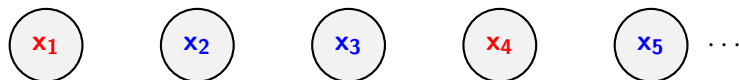
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DONE!

Variants Of The Infinite Ramsey Theorem

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This is easy to prove using the same technique we used for the $c = 2$ case.

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The Finite Ramsey Thm for 2-Hypergraphs:

Thm For all k there exists $n = R(k)$ such that for all COL: $\binom{[n]}{2} \rightarrow [2]$ there exists a homog set of size k .