BILL, RECORD LECTURE!!!!

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The Large Can Ramsey Theorem

Exposition by William Gasarch

April 8, 2026

Large Can Ramsey

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We will use the **Inf Can Ramsey Theory** to get a contradiction.

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So what to do?

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The Correct Infinite to Finite Proof

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Key $\mathrm{COL}(e_2) \in \{1, 2\}$.

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Hence the proof by contradiction is over.

The Hopefully Correct Proof using 4-Ary Ramsey

Exposition by William Gasarch

April 8, 2026

Attempt at Proof of Large Can Ramsey Thm

We use COL to define $\mathrm{COL}' : \binom{\{k,k+1,\ldots,k+n}{2} \to [16]$
We then apply **Large 4-ary Ramsey Theorem.**

In the slides below $x_1 < x_2 < x_3 < x_4$.

Pairs that begin the same way are same color

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Each pair is either equal or not equal. So 2^{15} colors.