BILL, RECORD LECTURE!!!!

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The Infinite Can Ramsey Thm: Mileti's Proof

William Gasarch-U of MD

Hungarian Math Comp Problem

From the 1950 "Kürschák/Eötvös Math Competition":

There are 1950 cans of paint. Find an x *such that (1) there are either* x *cans of paint all the same color, or* x *cans of paint that are all different colors and (2) it is possible to have neither* $x+1$ *cans that are all the same nor* $x+1$ *cans that are all different.*

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Do the following with your group.

1) Solve the problem, so find the x and prove that it is correct.

2) Solve this generalization:

There are n cans of paint. Find an $f(n)$ such that (1) there are either x cans of paint all the same color, or $f(n)$ cans of paint that are all different colors and (2) it is possible to have neither $f(n)+1$ cans that are all the same nor $f(n)+1$ cans that are all different.

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3) If there are ≤ 44 diff colors and each color appears ≤ 44 times then $\le 44 * 44 = 1936 < 1950$ cans.

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1) If there are 45 different paint colors DONE
2) If there are 45 of the same color then DONE
3) If there are ≤ 44 diff colors and each color appears ≤ 44 times then $\le 44 * 44 = 1936 < 1950$ cans.
4) CAN have NEITHER 46 the same NOR 46 different:
Color 1st 45 1, 2nd 45 2, ..., 43nd 45 43. You've colored $43 \times 45 = 1935$. Color the rest 44. Have used 44 colors.

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1) If there are 45 different paint colors DONE
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Color 1st 45 1, 2nd 45 2, ..., 43nd 45 43. You've colored $43 \times 45 = 1935$. Color the rest 44. Have used 44 colors.

For the more general problem $f(n)$ is roughly $\sqrt{n}$. I leave it to you to work out floors, ceilings, ± 1 stuff.

Can Ramsey Thm

The Can Ramsey Thm is for any number of colors.

It is named **Can Ramsey** in honor of the paint can problem on the 1950 Kürschák/Eötvös Math Competition

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Do not necessarily get a homog set since could color EVERY vertex differently. But then get infinite **rainbow set**.

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Thm: Let V be a countable set. Let $\mathrm{COL} : V \to \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).

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Prove with your neighbor.

Ramsey's Thm For Graphs

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Conjecture

Conjecture For every $\mathrm{COL} : \binom{\mathsf{N}}{2} \rightarrow \omega$ there is an infinite homog set OR an infinite rainb set.

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Get into groups and try to either PROVE or DISPROVE the Conjecture.

Conjecture is . . .

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FALSE:

- $\mathrm{COL}(i, j) = \min\{i, j\}$.
- $\mathrm{COL}(i, j) = \max\{i, j\}$.

Min-Homog, Max-Homog, Rainbow

Def: Let $\mathrm{COL} : \binom{\mathsf{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathsf{N}$. Assume $a < b$ and $c < d$.

- V is *homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff *TRUE*.
- V is *min-homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff $a = c$.
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Can Ramsey Thm for $\binom{\mathsf{N}}{2}$: For all $\mathrm{COL} : \binom{\mathsf{N}}{2} \rightarrow \omega$, there exists an infinite set V such that either V is homog, min-homog, max-homog, or rainb.

History

Year	Authors	Used
1952	Erdos & Rado	4-hypergraph Ramsey
1986	Rado	3-hypergraph Ramsey
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Our concern is educational.

First Step of Our Construction

Look at 1 and all of the edges coming out of it:

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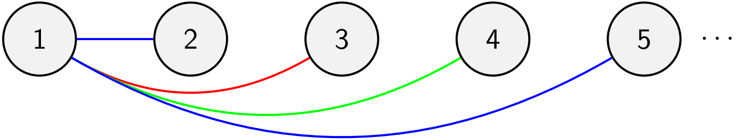

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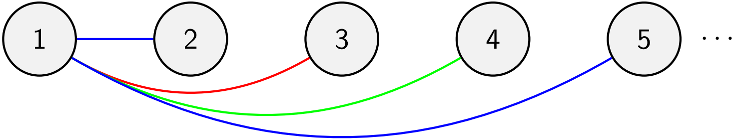

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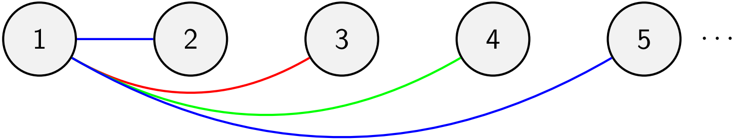

Either
1) Some color comes out of node 1 infinitely often.
2) No color comes out of node 1 infinitely often.

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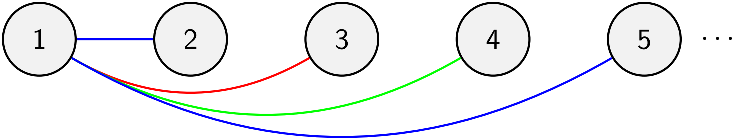

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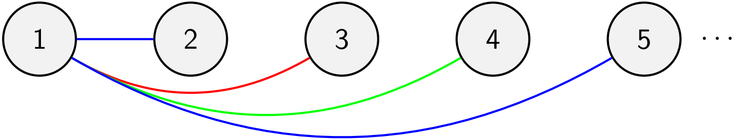

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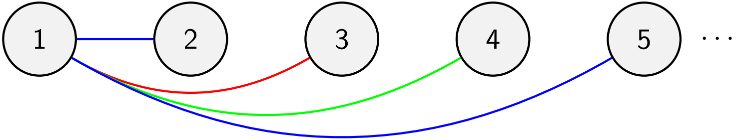

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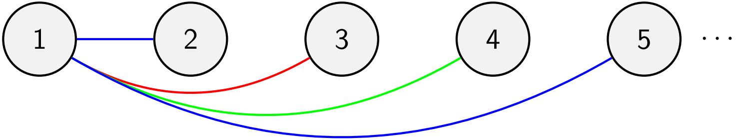

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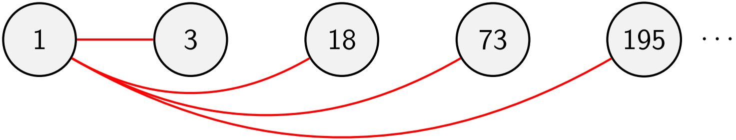

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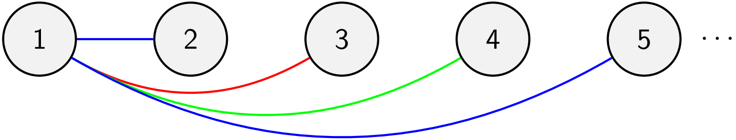

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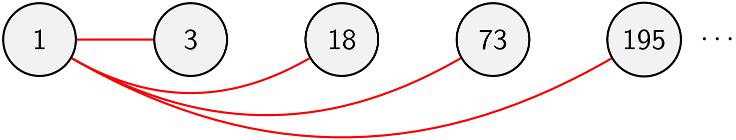

Set $x_1 = 1$ and $\mathrm{COL}'(x_1) = c_1$ (RED).

Case 2: No Color Comes Out of Node 1 Inf Often

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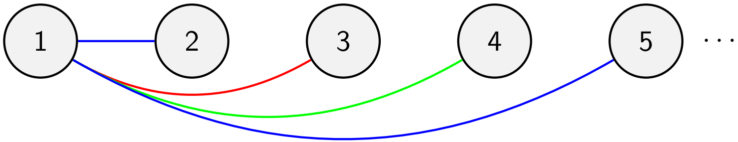

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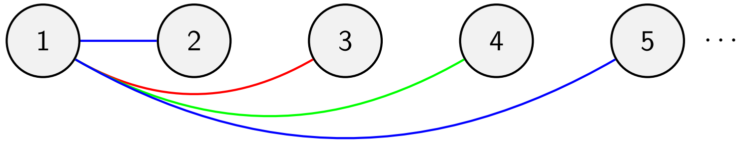

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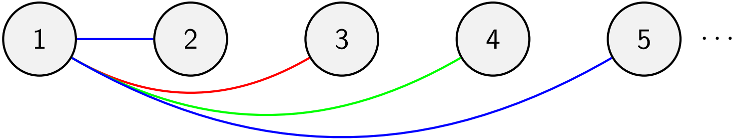

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Kill all those who dare to repeat a color!

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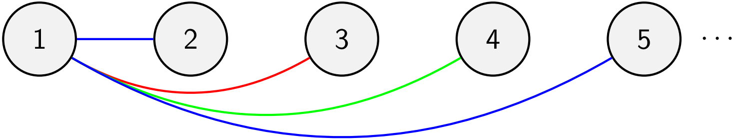

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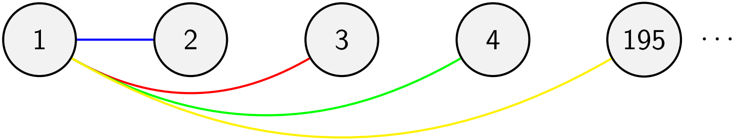

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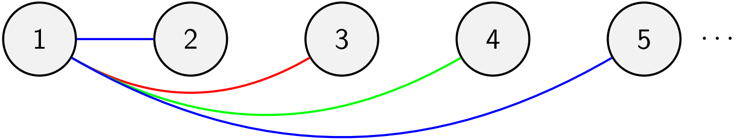

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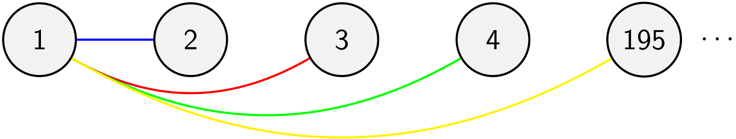

Set $x_1 = 1$ and $\mathrm{COL}'(x_1) = \mathrm{RB}_1$ (Rainbow 1).

The Next Step

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Let x_1 be 1 and x_2 be the least node that survives.

Easy Case 1 $\exists c$ x_2 has infinite number of c's coming out of it. Kill all those who disagree and set $\mathrm{COL}'(x_2) = c$.

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Next slide is a picture of this.

Node 1 and Node 2

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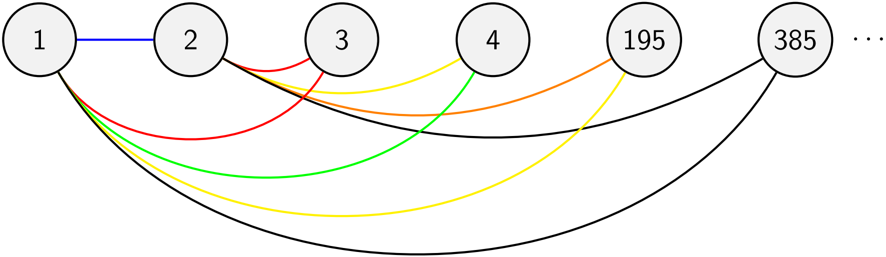

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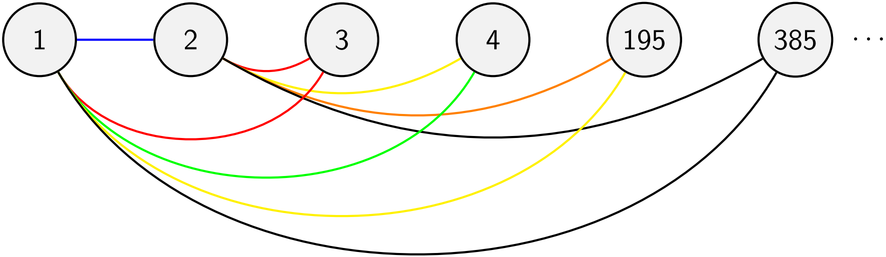

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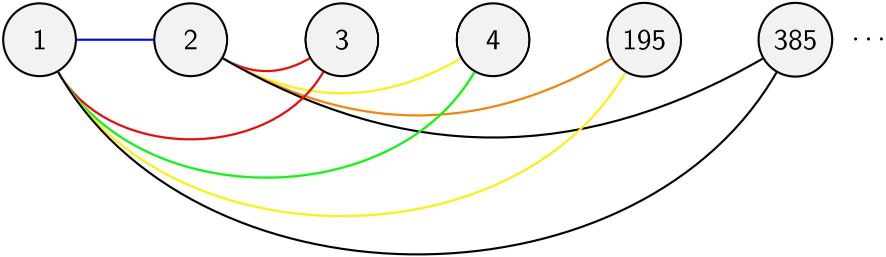

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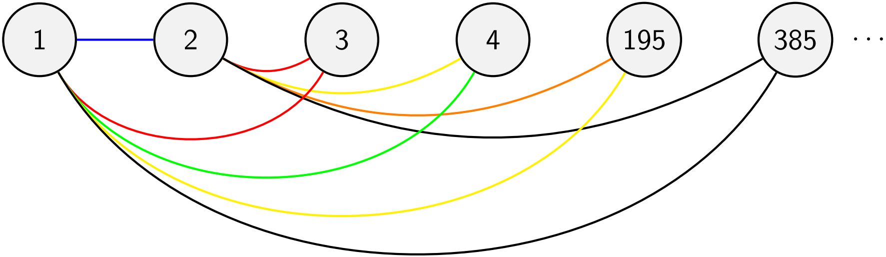

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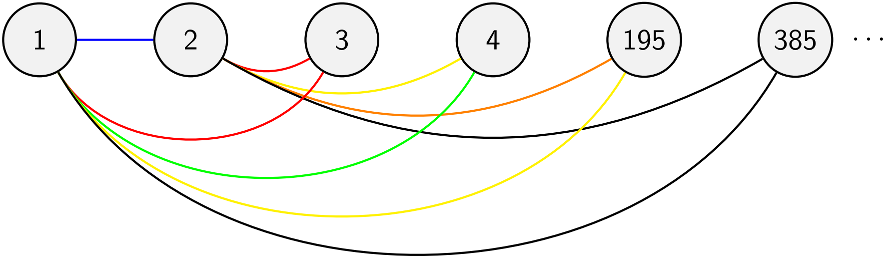

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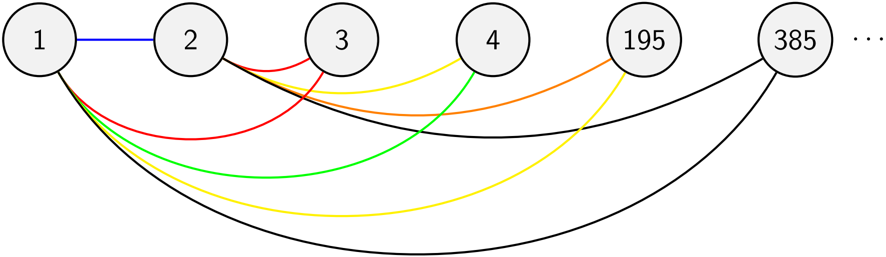

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1 and 2 **agree on 385** since $\mathrm{COL}(1,385) = \mathrm{COL}(2,385)$.
AH $\exists^\infty y$ 1 & 2 agree on y, OR finite Num of y, 1 and 2 agree on.

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Kill all those where 1 and 2 disagree.

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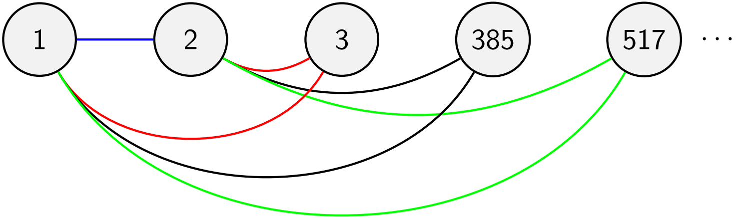

1 And 2 Agree Only Finitely Often

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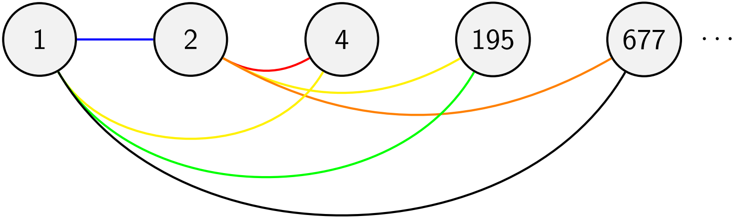

Recap: Where Are We Now

We have
$x_1, x_2 \in N$, $x_1 < x_2$, and $V_1, V_2 \subseteq N$, $V_1 \supseteq V_2$ are infinite.

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$x_1, x_2 \in \mathsf{N}$, $x_1 < x_2$, and $V_1, V_2 \subseteq \mathsf{N}$, $V_1 \supseteq V_2$ are infinite.

If $\mathrm{COL}'(x_1) = c_1$, a color then $(\forall y \in V_1)[\mathrm{COL}(x_1, y) = c_1]$.
If $\mathrm{COL}'(x_1) = \mathrm{RB}_1$ then $(\forall y, z \in V_1)[\mathrm{COL}(x_1, y) \neq \mathrm{COL}(x_1, z)]$.

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If x_1 and x_3 **agree infinitely often** then make them always agree and set $\mathrm{COL}'(x_3) = \mathrm{RB}_1$. DONE with step 3.

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Might be a HW to make this into a formal contruction.

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$\mathrm{COL}' \colon X \to \omega \cup \{\mathrm{RB}_i \colon i \in \omega\}$.

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where $\mathrm{COL}'(h_i) = c_i$ with c_i's different.

Example of Case 2

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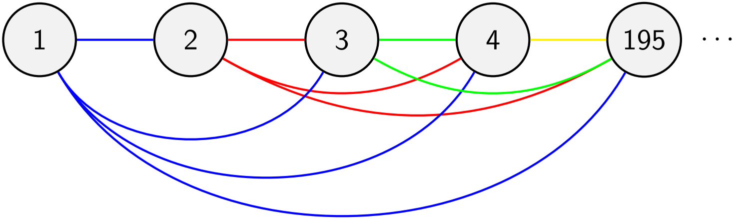

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$c_i = c_k$ iff $i = k$ iff $h_i = h_k$.

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For Cases 3,4 assume $(\forall x \in X)(\exists i)[\mathrm{COL}'(x) = \mathrm{RB}_i]$.

ωth Step, Case 3

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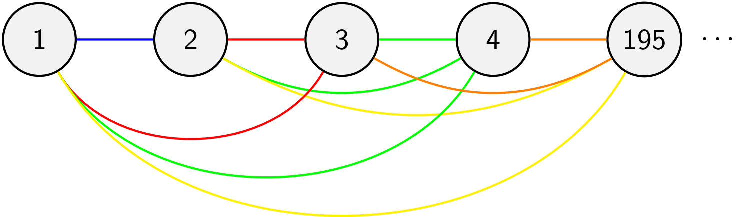

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I'll skip this but it might be a HW.

ωth Step, Case 4

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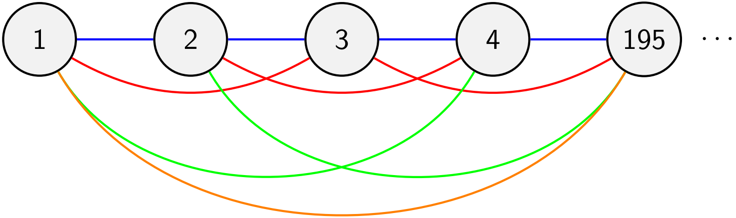

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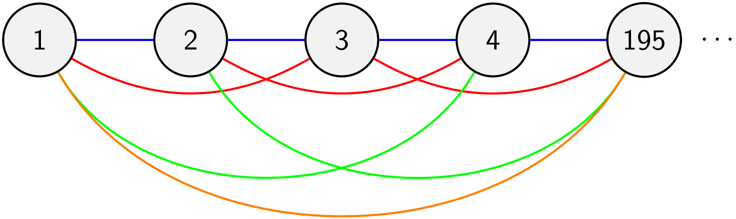

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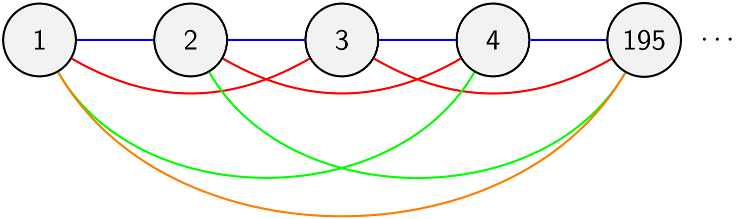

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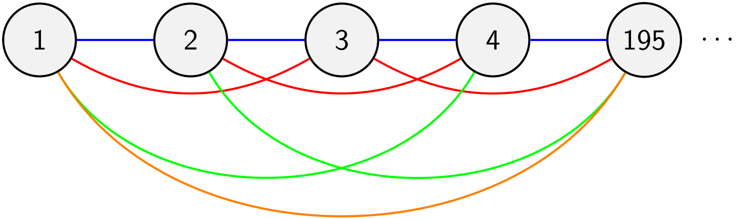

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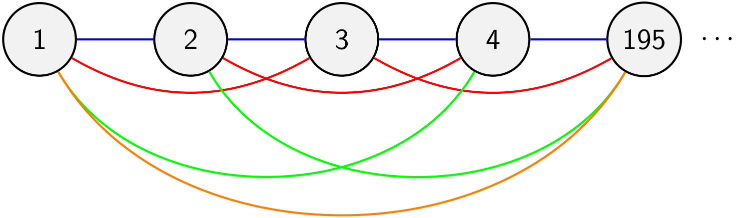

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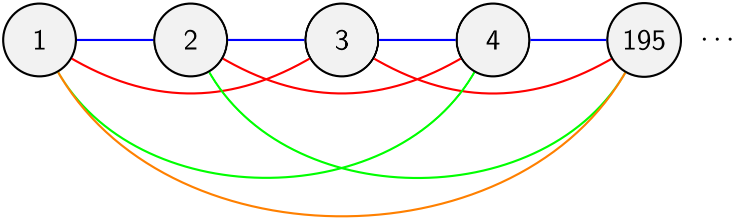

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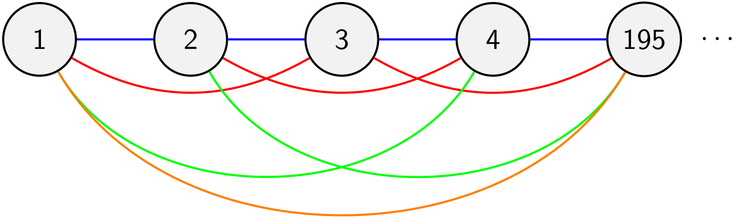

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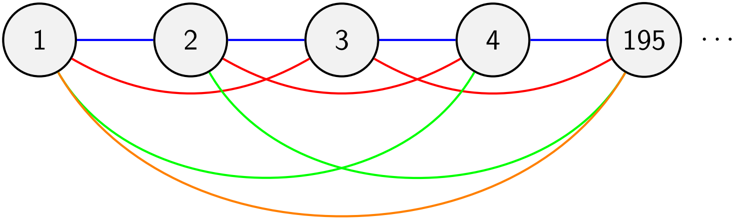

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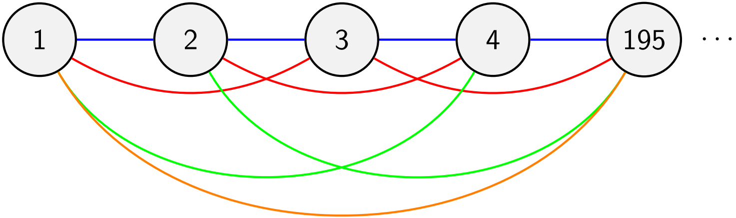
We prove a general theorem about such graphs.

A Theorem About Colored Graphs

Thm Let X be infinite. Let $\mathrm{COL} : \binom{X}{2} \rightarrow \omega$. Assume that, for every $x \in X$ and $c \in \omega$, $\deg_c(x) \leq 2$. Then there exists an infinite rainbow $X_1 \subseteq X$.

Examples



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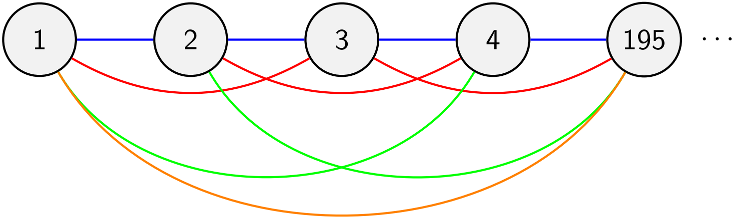

Work in Groups

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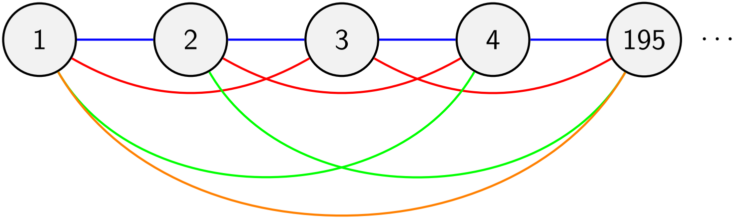

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Find an infinite rainbow subset of this graph.

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If you keep taking vertices further and further apart you get a rainbow set.

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The general construction is similar: We take vertices far out.

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