BILL, RECORD LECTURE!!!!

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Primitive Recursive Functions

Exposition by William Gasarch-U of MD

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4. $g_1(x_1,\ldots,x_k)$, …, $g_n(x_1,\ldots,x_k)$,$h(x_1,\ldots,x_n)$ PR $\implies$

$$f(x_1,\ldots,x_k) = h(g_1(x_1,\ldots,x_k),\ldots,g_n(x_1,\ldots,x_k)) \text{ is PR}$$

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4. $g_1(x_1, \ldots, x_k)$, ..., $g_n(x_1, \ldots, x_k)$,$h(x_1, \ldots, x_n)$ PR $\Longrightarrow$

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5. $h(x_1, \ldots, x_{n+1})$ and $g(x_1, \ldots, x_{n-1})$ PR $\Longrightarrow$

$$\begin{aligned} f(x_1, \ldots, x_{n-1}, 0) &= g(x_1, \ldots, x_{n-1}) \\ f(x_1, \ldots, x_{n-1}, m+1) &= h(x_1, \ldots, x_{n-1}, m, f(x_1, \ldots, x_{n-1}, m)) \end{aligned}$$

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The PR functions can be put in a hierarchy depending on how many times the recursion rule is used to build up to the function.

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Used Rec Rule five times.
What should we call this? Discuss

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Its been called WOWER (in Graham-Rothchild-Spencer Ramsey Theory Book).

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f_6 and beyond have no name.

Levels

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Note One can show that any finite number of exponentials is in PR_3.

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7. $f(x) = 1$ if x is the sum of 2 primes, 0 otherwise.

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Discuss.

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Yes. We will see a contrived one on the next slide.

A Contrived Not PR Function

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Let $f_1, f_2, \ldots$ be all of the PR functions.

$$F(x) = f_x(x) + 1$$

is computable but not a PR function.

A “Natural” non PR Function

Def **Ackermann’s function** is the function defined by

$$
\begin{aligned}
A(0,y) &= y+1\\
A(x+1,0) &= A(x,1)\\
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3. A grows faster than any PR function.
4. Since A is defined using a recursion which involves applying the function to itself there is no obvious way to take the definition and make it PR. Not a proof, an intuition.

Ackermann’s Function is Natural: Security

https://www.ackermansecurity.com/

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They are called Ackerman Security since they claim that a thief would have to take time Ackerman(n) to break in.

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So $nA^{-1}(n,n)$ is the exact upper and lower bound!

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1. Goodstein Sequences (next slide packet).
2. Finite Version of Kruskal's Tree Theorem.