

# BILL, RECORD LECTURE!!!!

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# Are There Better Bounds on the VDW Numbers?

**Exposition by William Gasarch**

April 21, 2026

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**Logician (Shelah) proved  $W(k, c)$  prim rec: clever!**

- ▶ Proof is elementary. Can present here but won't.
- ▶ Bounds still large. Fifth Level of PR hierarchy.

# Deep Math From Search for Better Upper Bounds on VDW Numbers

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It DID succeed! (Oh! Thats a good thing!)

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## Examples

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1. For all  $k$ ,  $\{x : x \equiv 0 \pmod{k}\}$  has upper den  $\frac{1}{k}$ .
2.  $\{x^2 : x \in \mathbb{N}\}$  has upper den 0.

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The hope was that the proof of Conj would require a new proof of VDW's Theorem that would lead to better bounds.

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- ▶ Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).

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  - ▶ Combinatorics was less respected in 1975 than in 1998.
  - ▶ Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.

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None of these results used mathematics of interest.

# Known Lower Bounds

1. Easy Use of Prob Method  $W(k, 2) \geq \sqrt{k}2^{k/2}$  (Easy extension to 3 colors)
2. Very sophisticated use yields  $W(k, 2) \geq \frac{2^k}{k^\epsilon}$  (Does not extend to 3 colors.)
3. If  $p$  is prime then  $W(p, 2) \geq p(2^p - 1)$ . Constructive! (Does not extend to 3 colors.)

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