

BILL, RECORD LECTURE!!!!

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Van Der Warden's (VDW) Thm

Exposition by William Gasarch

April 21, 2026

VDW's Thm

Def Let $W, k, c \in \mathbb{N}$. Let $\text{COL}: [W] \rightarrow [c]$. A **mono k -AP** is an arithmetic progression of length k where every elements has the same color. We often say

$a, a + d, \dots, a + (k - 1)d$ are all the same color

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$W(3, 2)$ exists

We will determine W later.

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1. Within one block.
2. If we take enough blocks, how they relate.

Within a Block

Def: $a, a + d, a + 2d$ is an **almost mono 3AP** if $\text{COL}(a) = \text{COL}(a + d) \neq \text{COL}(a + 2d)$. The **color of an almost mono 3AP** is $\text{COL}(a) = \text{COL}(a + d)$.

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Look at the first three elements of a block of 5:

1. **RRR** or **BBB**. 1-2-3 is mono 3AP.
2. **RBR** or **BRB**. 1-3-5 is mono 3AP or almost mono 3AP.
3. **RBB** or **BRR**. 2-3-4 is mono 3AP or almost mono 3AP.
4. **BBR** or **RRB**. 1-2-3 is almost mono 3AP.
5. **BRB**. 1-3-5 is a mono 3AP or an almost mono 3AP.
6. **BRR**. 2-3-4 is mono 3AP or almost mono 3AP.

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Hence need to take $W = 5 \times 65 = 365$.

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We can get by with LESS blocks- we will consider this point after the proof.

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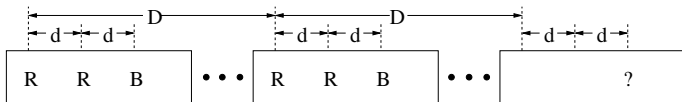
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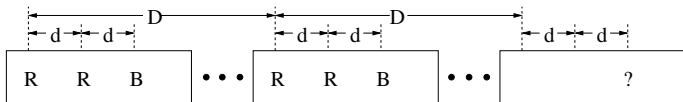


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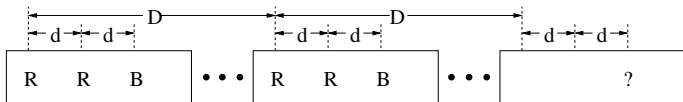
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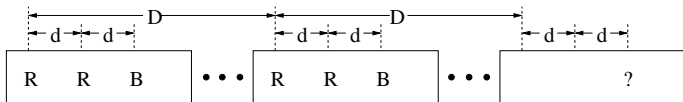
If ? is **R** then get **R** 3-AP.

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If ? is **B** then get **B** 3-AP.

If ? is **R** then get **R** 3-AP.

Done!

Side Note: Can Get By With Less Blocks

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How many colorings of a block already have a mono 3AP.

Side Note: Can Get By With Less Blocks (cont)

RRRXY with $X, Y \in \{R, B\}$. 4 colorings.

BBBXY with $X, Y \in \{R, B\}$. 4 colorings.

RBRRR

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RBBBX with $X \in \{R, B\}$. 2 colorings.

BRRRX with $X \in \{R, B\}$. 2 colorings.

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Side Note: Can Get By With Less Blocks (cont)

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There are 16 blocks which already have a mono 3AP. Hence can use $32 - 16 = 16$ blocks.

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I really do not care.

Is $W(3, 2) = 365$?

No

What is $W(3, 2)$?

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One can work out by hand that

$$W(3, 2) = 9.$$

We will later say which VDW numbers are known and how they compare to the bounds given by the proof of VDW's Thm.

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Spoiler Alert The few known VDW numbers are **much smaller** than the bounds given by the proof of VDW's Thm.

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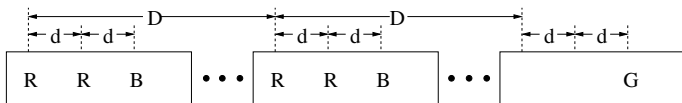
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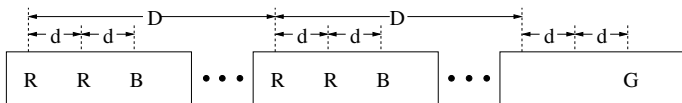
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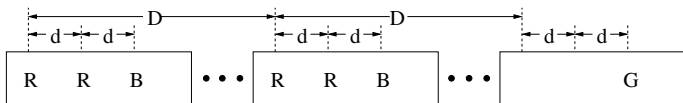
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We have 2 almost mono 3APs of diff colors that same last element.

I Like Big Blocks and I Cannot Lie!

Let W be LOTS of blocks of size $7 \times 2 \times (3^7 + 1)$.

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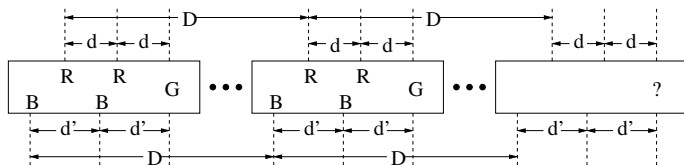
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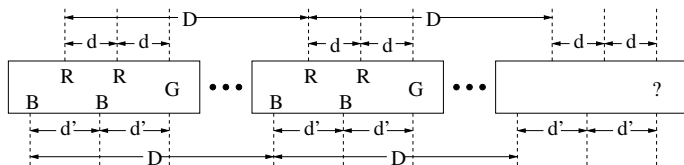


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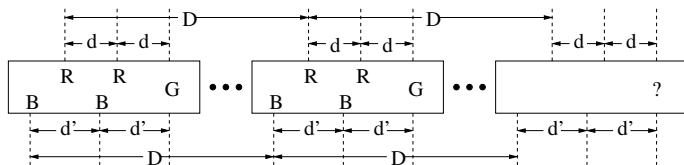
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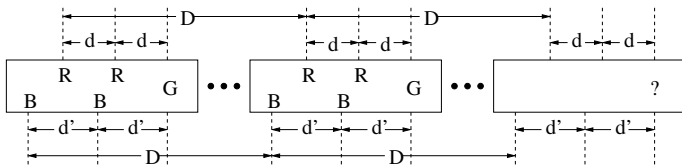
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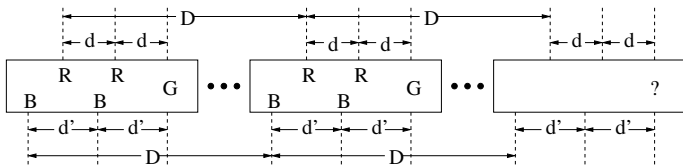
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- ▶ You can BELIEVE that $W(3, c)$ exists though might wonder how to prove it formally.
- ▶ There are ways to formalize the proof; however, they are not enlightening.

$W(3, c)$

From what you have seen:

- ▶ You COULD do a proof that $W(3, 4)$ exists. You would need to iterate what I did twice.
- ▶ You can BELIEVE that $W(3, c)$ exists though might wonder how to prove it formally.
- ▶ There are ways to formalize the proof; however, they are not enlightening.
- ▶ The Hales-Jewitt Thm is a general theorem from which VDW is a corollary. We won't be doing that.

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Note that we **do not** do

$W(3, 2) \implies W(3, 3)$.

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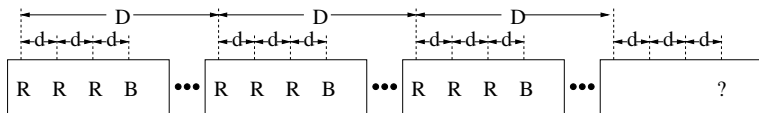
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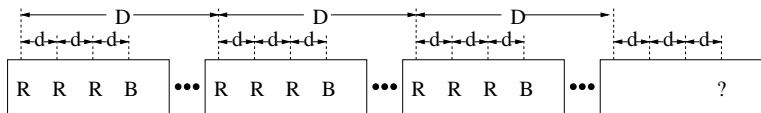
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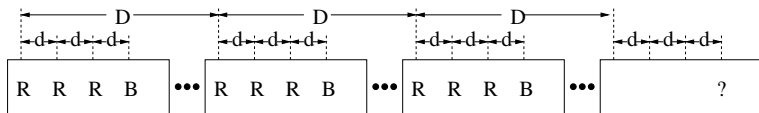
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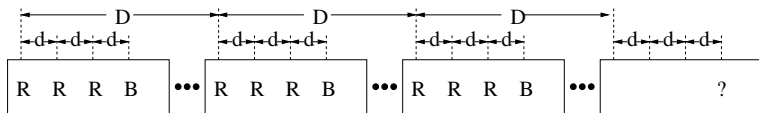
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Done!

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