BILL, RECORD LECTURE!!!!

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Well Quasi Orders: Fast Growing Functions

Exposition by William Gasarch

February 18, 2026

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We need to prove that $\mathrm{tree}(n)$ always exists.

For all n, $\mathrm{tree}(n)$ Exists

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We now form an ∞ bad sequence which is a contradiction.

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What is Known About *tree*(n)?

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What is Known About *tree*(*n*)?

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tree grows **much faster** than Ackermann's function.

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We can define more levels of a hierarchy by indexing functions by countable ordinals.

Lets look at the countable ordinals

$1, 2, 3, \ldots, \omega$

$\omega + 1, \omega + 2, \ldots, 2\omega$

$\omega, 2\omega, 3\omega, \ldots, \omega^2.$

$\omega, \omega^2, \omega^3, \ldots, \omega^\omega$

$\omega\omega^\omega, \omega^{\omega^\omega}, \ldots \epsilon_0$

$\mathrm{tree}(n)$ grows rougly like f_{ϵ_0}

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Known:
$\text{TREE}(1) = 1$ $\text{TREE}(2) = 3$ $\text{TREE}(3)$: See Next Slide.

tree vs TREE

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Suffice to say that $\mathrm{TREE}(n)$ grows much faster than $\mathrm{tree}(n)$.

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Is the function TREE natural? VOTE
Who says YES:
Who says NO:

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Known: $\mathrm{GM}(1)$ is not known.

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Known: $\mathrm{GM}(1)$ is not known.

GM grows much faster than **gm** and TREE.

How Come TREE is Better Known Than GM

I asked ChatGPT:
Why is $\mathrm{TREE}(3)$ better known than analogous $\mathrm{GM}(k)$ functions?
Short answer: $\mathrm{GM}(k)$-type functions are at least as enormous as $\mathrm{TREE}(k)$, but they are far less canonical, far less clean, and much harder to package into a single dramatic integer.

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[Bill Here] see next few slides for more ChatGPT commentary and my commentary on their commentary.

Simple vs Complicated Definitions

$\mathrm{TREE}(3)$ has a remarkably compact definition:
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[Bill Here] Agree. Note that I can teach KTT in this class but I could not teach GMT in any class.

Proof-Theoretic Position

$\mathrm{TREE}(3)$ sits at a beautifully sharp proof-theoretic boundary: Kruskal's theorem is not provable in Peano Arithmetic, and the associated growth is connected to ordinal analysis near Γ_0.

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[Bill Here] This may be why logicians prefer KTT to GMT but not why the public does.

Shock Value and Simplicity

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Graph minors has the incomprehensible magnitude but lacks that simplicity hook.

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We have nothing like that for gm or GM.