BILL, RECORD LECTURE!!!!

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Well Quasi Orders: Kruskal Tree Theorem and the Graph Minor Theorem

Exposition by William Gasarch

February 18, 2026

The Kruskal Tree Theorem

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The Kruskal Family should talk with each other more.

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Can use this to get $T_i \preceq T_j$. We are Done!

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Proof is similar to the proof of **Kruskal Tree Theorem**

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It may happen in Ilya Hajiaghayi's lifetime since he might do it.

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