BILL, RECORD LECTURE!!!!

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Set of all Finite Subsets

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WRITE DOWN this definition.

Examples

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$\{1, 2, 3, 4\}$	$\{2, 3, 4, 5\}$	

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$\{1,2,3,100\}$	$\{10,000,000\}$	

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$\{4,7,9,100\}$	$\{1,\ldots,100\}$	

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$\{4,7,9,100\}$	$\{1,\ldots,100\}$	$A \preceq B$
$\emptyset$	$\{1\}$	$A \preceq B$

Today's Theorem

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We will prove the following which you should **WRITE DOWN**

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Theorem $(2^{\text{fin}\mathbb{N}}, \preceq)$ is a wqo.

What Do We Mean By Smallest?

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take the smallest element that blah

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All of our elements are **finite sets**.

Smallest means the least cardinality.

If there is a tie then pick one arbitrarily

Proof that $2^{\mathrm{fin}\mathbb{N}}$ is a wqo: Bad Seqs

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Thm $(2^{\text{fin}\mathbb{N}}, \preceq)$ is a wqo.

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Def A **bad sequence** is $X_1, X_2, \ldots$ with no uptick.
(Recall that the X_i are all finite subsets of $\mathbb{N}$.)

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Write Down the definition of a bad sequence.

$2^{\mathrm{fin}\mathbb{N}}$ is a wqo: Min Bad Seqs. Step 1

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$\{3, 5, 10, 12\}, \{10, 12, 33\}, \{1, 90\}, \ldots$ (no pattern implied)

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Thin out **BAD** by eliminating all sequences in **BAD** that don't begin with B_1.

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B_2 is the smallest set that begins a sequence in **BAD**.

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Write Down Def of new **BAD** and B_2.

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The sequence $B_1, B_2, \ldots,$ is a bad sequence (easy).

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Write Down
B_i is the smallest (or tied) ith elt of a bad seq that begins with
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Shaving The Min Bad Sequence

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Let b_i be the least element in B_i.
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Example

If $B_1 = \{10, 111, 900\}$ then $b_1 = 10$ and $C_1 = \{111, 900\}$.

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Let $XX = \{C_1, C_2, C_3, \ldots\}$.

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Let $XX = \{C_1, C_2, C_3, \ldots\}$. This is a **set** not a **sequence**.

Write Down The definition of b_i and C_i and XX.

We Prove $(XX, \preceq)$ is a wqo!

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$C_{99}, C_{78}, C_{18}, C_{99}, C_{70}, C_{87231}, \ldots$ (No pattern implied).

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$C_{18}, C_{99}, C_{70}, C_{87231}, \ldots$ is also a bad sequence.

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Upshot Can assume that the first index is the least index in the sequence.

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Write Down First index is the least index in the sequence.

We Keep Proving $(XX, \preceq)$ is a wqo!

Assume there is a bad sequence that begins with C_{18} and that this is the least index that appears:

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Look at the sequence

$B_1, B_2, \ldots, B_{17}, C_{18}, C_{99}, C_{70}, C_{87231}, \ldots$ (No pattern implied).

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Assume there is a bad sequence that begins with C_{18} and that this is the least index that appears:

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Look at the sequence

$B_1, B_2, \ldots, B_{17}, C_{18}, C_{99}, C_{70}, C_{87231}, \ldots$ (No pattern implied).

This is

A bad sequence that begins with $B_1, B_2, \ldots, B_{17}$ and whose next element is C_{18}.

We Keep Proving $(XX, \preceq)$ is a wqo!

Assume there is a bad sequence that begins with C_{18} and that this is the least index that appears:
$C_{18}, C_{99}, C_{70}, C_{87231}, \ldots$ (No pattern implied).

Look at the sequence

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But $|C_{18}| < |B_{18}|$. So C_{18} is smaller than B_{18} contradicting def of B_{18}.

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Write Down XX is a wqo.

Finish the Proof that $2^{\text{fin}\mathbb{N}}$ is a wqo

Finish the Proof that $2^{\mathrm{fin}\mathbb{N}}$ is a wqo

We have XX is a wqo.

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Hence $XX \times \mathbb{N}$ is a wqo.

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Hence $2^{\mathrm{fin}\mathbb{N}}$ is a wqo. We are Done!

Recap of Proof that $2^{\text{fin}\mathbb{N}}$ is a wqo

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Assume, BWOC, that $2^{\mathrm{fin}\mathbb{N}}$ is not a wqo.

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g shows $B_i \preceq B_j$. Contradicts $B_1, B_2, \ldots$ being a bad sequence.

Finite Version

Finite Version

Bad sequences are finite.

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Is there a bad sequence of length 10^{100}?

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What is the largest L such that there is a bad sequence $A_1, A_2, \ldots, A_L$ where $\max(A_i) \leq i$.

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There is an easy proof and an interesting proof that generalizes. We do the interesting proof.

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We now form an ∞ bad sequence which is a contradiction.

Look at A_{11}, A_{21}, ... These are all subsets of $\{1, \ldots, 11\}$

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Kill all the sequences that do not begin with B_1.

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Look at the second elements of the sequences that are left.

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Some set occurs ∞ often. Let B_1 be that set.
Kill all the sequences that do not begin with B_1.
Look at the second elements of the sequences that are left.
There are all subsets of $\{1, \ldots, 12\}$.

Finite Version

Does there exist L such that there is a bad sequence $A_1, A_2, \ldots, A_L$ where $\max(A_i) \le i + 10$.

Assume, BWOC, that for all L, there is a bad sequence $A_1, A_2, \ldots, A_L$ where $\max(A_i) \le i + 10$.

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We now form an ∞ bad sequence which is a contradiction.

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There will be other wqo where a similar f grows much faster.