BILL, RECORD LECTURE!!!!

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Well Quasi Orders

Exposition by William Gasarch

February 18, 2026

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5) wqo theory leads to a surprising result in **automata theory**.

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Formally, for all i, j, $x_i \not\preceq x_j$ and $x_j \not\preceq x_i$.

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No $\infty \downarrow$ seq.
There is an ∞ antichain: $2, 3, 5, 7, 11, 13, 17, \ldots$ (The primes)

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WORK IN GROUPS Is there an infinite antichain?

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Note Not just contradicts. All we needed was some (x_i, y_i) compared to some (x_j, y_j). We got an $\infty \uparrow$ seq!

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$2^{\mathbb{N}}$ and $2^{\text{fin}\mathbb{N}}$ have an ∞ antichain: $\{1,2\}$, $\{1,3\}$, $\{1,4\} \cdots$

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Note We show that $(X, \preceq)$ is a **wqo** iff $(X, \preceq)$ is **javier**.

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So color is UP.

Bonus We don't just get an uptick, we get an $\infty \uparrow$ sequence.

wqo $\implies$ javier

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Since the sequence has an uptick it **is not** $\infty \downarrow$ and not an ∞ antichain.

Summary of javier, wqo, and $\infty \uparrow$

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2. For every sequence $x_1, x_2, \ldots$ there is an increasing subsequence.
3. No sequence $x_1, x_2, \ldots$ is an $\infty \downarrow$ or an ∞ anti-chain.

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Proof: This is on the HW.

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Then $(2^{\mathrm{fin} X}, \preceq')$ is a wqo.

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