Canonical Ramsey's Theorem (finite version)
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1 Introduction

We present the best proof of Can Ramesy for Graphs in terms of bounds.

We will need the one-dom caseof it:

Lemma 1.1 *Let COL be any coloring of $[(m-1)^2+1]$. Then there exists either a homog set of size m or a rainbow set of size m.*

2 A Premise that Yields a Rainbow Set

The next definition and lemmas gives a way to get a rainbow set under some conditions.

Def 2.1 Let COL be a coloring of $\binom{[m]}{2}$. If c is a color and $v \in [m]$ then $\deg_c(v)$ is the number of c-colored edges with an endpoint in v.

The following theorem is due to Babai [2]. We include the proof since the paper is not available on-line and will eventually be lost to history.

Lemma 2.2 *Let $m \geq 3$. Let COL be a coloring of $\binom{[m]}{2}$. If for all $v \in [m]$ and all colors c $\deg_c(v) \leq 1$ then there exists a rainbow set of size $\geq (2m)^{1/3}$.*

Proof:

Let X be a maximal rainbow set. This means that,

$$(\forall y \in [m] - X)[X \cup \{y\} \text{ is not a rainbow set}].$$

Let $y \in [m] - X$. Why is $y \notin X$? One of the following must occur:

1. There exists $u, u_1, u_2 \in X$ such that $u_1 \neq u_2$ and $COL(y, u) = COL(u_1, u_2)$. (It is possible for $u = u_1$ or $u = u_2$.)

2. There exists $u_1 \neq u_2 \in X$ such that $COL(y, u_1) = COL(y, u_2)$. This cannot happen since then y has some color degree ≥ 2.

We map $[m]-X$ to $X \times \binom{X}{2}$ by mapping $y \in [m]-X$ to $(u, \{u_1, u_2\})$ as indicated in item 1 above. This map is injective since if y_1 and y_2 both map to $(u, \{u_1, u_2\})$ then $COL(y_1, u) = COL(y_2, u)$.

This map has domain of size $n - |X|$ and co-domain of size $|X|\binom{|X|}{2}$. Hence

$$m - |X| \le |X|\binom{|X|}{2} = |X|^2(|X|-1)/2 = \frac{|X|^3 - |X|^2}{2} \le \frac{|X|^3}{2} - |X|$$

$$m \le \frac{|X|^3}{2}.$$

$$|X| \ge (2m)^{1/3}.$$

▮

Alon, Lefmann, and Rodl [1] have obtained a slight improvement and also showed that it cannot be improved past that.

Lemma 2.3 *Let $m \ge 3$.*

1. *Let COL be a coloring of $\binom{[m]}{2}$. If for all $v \in [m]$ and all colors c, $\deg_c(v) \le 2$ then there exists a rainbow set of size $\ge \Omega((m \log m)^{1/3})$.*

2. *There exists a coloring of $\binom{[m]}{2}$. such that for all $v \in [m]$ and all colors c, $\deg_c(v) \le 1$ and all rainbow sets are of size $\le O((m \log m)^{1/3})$.*

3 Main Theorem

We remind the reader of the definition of $\deg_c$ and also add the definitions of $\deg_c^L$ and $\deg_c^R$.

Def 3.1 Let COL be a coloring of $\binom{[m]}{2}$. Let c be a color and let $v \in [m]$

1. $\deg_c^R(v)$ is the number of c-colored edges (v, u) with $v < u$.

2. $\deg_c^R(v)$ is the number of c-colored edges (v, u) with $u < v$.

3. A *bad triple* is a triple a, b, c such that a, b, c does not form a rainbow K_3.

The next two lemmas show us how to, in some cases, reduce the number of bad triples.

Lemma 3.2 *Let COL be a coloring of $\binom{[m]}{2}$ such that, for every color c and vertex v, $\deg_c(v) \le d$. Then the number of bad triples is less than $\frac{dm^2}{2}$.*

Proof: Let b be the number of bad triples. We upper bound b by summing over all v that are the point of the triple with two same-colored edges coming out of it.

$$b \le \sum_{v\in[m]} \sum_{c\in \mathsf{N}} \text{Num of bad triples } \{v,u_1,u_2\} \text{ with } COL(v,u_1) = COL(v,u_2) = c\ .$$

(Note that we are not assuming $v < u_1, u_2$.)

We bound the inner summation. Since v is of degree $m-1$ we can renumber the colors as $1, 2, \ldots, m-1$ (some of the $\deg_c(v)$ may be 0). Hence

$$b \le \sum_{v\in[m]} \sum_{c=1}^{m-1} \binom{\deg_c(v)}{2}.$$

Note that $\sum_{c=1}^{m} \deg_c(v) = m-1 \le m$ and $(\forall c)[\deg_c(v) \le d]$. The inner sum is maximized when $d = \deg_1(v) = \deg_2(v) = \cdots = \deg_{m/d}(v)$ and the rest of the $\deg_c(v)$'s are 0. Hence we have

$$b \le \sum_{v\in[m]} \sum_{c=1}^{m} \binom{\deg_c(v)}{2} \le \sum_{v\in[m]} (m/d)\binom{d}{2} < m\frac{m}{d}\frac{d^2}{2} = \frac{dm^2}{2}.$$

▮

Lemma 3.3 *Let COL be a coloring of $\binom{[m]}{2}$ that has b bad triples. Let $1 \le m' \le m$. There exists an m'-sized set of vertices with $\le b\left(\frac{m'}{m}\right)^3$ bad triples.*

Proof: Pick a set X of size m' at random. Let E be the expected number of bad triples. Note that

$$E = \sum_{\{v_1,v_2,v_3\}\text{ bad}} \text{Prob that } \{v_1,v_2,v_3\} \subseteq X\ .$$

Let $\{v_1, v_2, v_3\}$ be a bad triple. The probability that all three nodes are in X is bounded by

$$\frac{\binom{m-3}{m'-3}}{\binom{m}{m'}} = \frac{m'(m'-1)(m'-2)}{m(m-1)(m-2)} \le \left(\frac{m'}{m}\right)^3.$$

Hence the expected number of bad triples is $\le b(\frac{m'}{m})^3$. Therefore there must exist some X that has $\le b(\frac{m'}{m})^3$ bad triples.

∎

Note 3.4 The above theorem presents the user with an interesting tradeoff. She wants a large set with few bad triples. If m' is large then you get a large set, but it will have many bad triples. If m' is small then you won't have many bad triples, but m' is small. We will need a Goldilocks-m' that is just right.

Now we can prove the theorem!

Theorem 3.5 *For all k the following hold.*

1.
$$ER(k) \le \left(\frac{k^9}{16}\right)^{2(k-2)^2+1} \le 2^{18k^2 \lg(k)}.$$

2.
$$ER(k) \le \left(\frac{9k^6}{16}\right)^{2(k-2)^2+1} \le 2^{12k^2 \lg(k)}.$$

Proof:

We will determine n later. We will have parameters m, m', m'', δ, s, t which we will choose later.

Intuition: In the usual proofs of Ramsey's Theorem we take a vertex v and see which of such that $\deg^R_{RED}(v)$ or $\deg^R_{BLUE}$ is large. One of them must be at least half of the size of the vertices still in play. Here we change this up:

- Instead of taking a particular vertex v we ask if there is *any* v and any color c such that either $\deg^L_c(v)$ or $\deg^R_c(v)$ is large. We hope to do this until either we have $(k-2)^2+1$ elements that have a large $\deg^L_c(v)$

for some c, or $(k-2)^2+1$ elements that have a large $\deg_c^R(v)$ for some c. We will then apply Lemma 1.1. (We take care of the extra point we need a different way.) We will need to iterate this process at most $2(k-2)^2+1$ times.

- What is large? Similar to the proof of Ramsey's theorem it will be a fraction of what is left, a fraction δ which we will pick later. Unlike the proof of Ramsey's theorem δ will depend on k.

- In the proof of Ramsey's theorem we were guaranteed that one of $\deg_{RED}(v)$ or $\deg_{BLUE}(v)$ was large. Here we have no such guarantee. We may fail. In that case something else happens and leads to a rainbow set!

Formally the construction will only use the points $\{2, \ldots, n-1\}$ so that we will have available a point bigger than all the points we finally have or smaller than. We ignore this in the construction and the analysis but we will point it out when we need it.

CONSTRUCTION

Phase 1:

Stage 0:

1. $V_0^L = V_0^R = \emptyset$. The set V_0^L will be vertices such that the edges from them to all vertices to their Left are the same color. Similar for V_0^R.

2. $N_0 = [n]$. COL' is not defined on any points.

Stage i: Assume that V_{i-1}^L, V_{i-1}^R, and N_{i-1} are already defined.

1. If there exists $x \in N_{i-1}$ and c a color such that $\deg_c^R(x) \geq \delta N_{i-1}$ then do the following:

$$\begin{aligned} V_i^R &= V_{i-1}^R \cup \{x\} \\ V_i^L &= V_{i-1}^L \\ N_i &= \{v \in N_{i-1} : x < v \wedge COL(x,v) = c\} \\ x_i &= x \\ COL'(x_i) &= c \end{aligned}$$

Note that $|N_i| \geq \delta |N_{i-1}|$. If $|V_i^R| = (k-2)^2+1$ then goto Phase $2a$.

2. If there exists $x \in N_{i-1}$ and c a color such that $\deg_c^L(x) \geq \delta N_{i-1}$ then do the following:

$$\begin{aligned} V_i^R &= V_{i-1}^R \\ V_i^L &= V_{i-1}^L \cup \{x\} \\ N_i &= \{v \in N_{i-1} : v < x \wedge COL(x,v) = c\} \\ x_i &= x \\ COL'(x_i) &= c \end{aligned}$$

Note that $|N_i| \geq \delta |N_{i-1}|$. If $|V_i^L| = (k-2)^2 + 1$ then goto Phase $2b$.

3. If neither case 1 or case 2 holds then goto Phase $2c$.

End of Phase 1

Since we goto Phase 2 if either $|V_i^R| = (k-2)^2 + 1$ or $|V_i^L| = (k-2)^2 + 1$ we iterate the above process at most $2(k-2)^2+1$ times. Let $s = 2(k-2)^2+1$.

Phase 2a: Restrict COL' to V_i^R and apply Lemma 1.1 to obtain that one of the following occurs.

1. There is a a set $H' \subseteq V_i^R$, homog relative to COL', of size $k-1$. Recall that n has not been used at all. It is easy to see that $H = H' \cup \{n\}$ is homog relative to COL.
2. There is a a set $H' \subseteq V_i^R$, rainbow relative to COL', of size $k-1$. Recall that n has not been used at all. It is easy to see that $H = H' \cup \{n\}$ is min-homog relative to COL.

We need to be able to carry out the construction for s stages. Note that after s stages $|N_s| \geq \delta^s n$. We need this to be ≥ 1. Hence we need

FIRST CONSTRAINT:

$$\delta \geq \left(\frac{1}{n}\right)^{1/s}.$$

If you got to Phase 2a you do NOT need to goto Phase 2b or 2c.

End of Phase 2a:

Phase $2b$**:** You got here because $|V_i^L| = (k-2)^2+1$. This is similar to Phase $2a$ so we omit it. We note that in this case you obtain either a homog set or a max-homog set.

End of Phase 2b:

Phase $2c$**:**

Assume that when you got here $N = N_i$ was of size m. The largest stage this could happen at was $s-1$. Hence we need

SECOND CONSTRAINT:

$$m \leq \delta^{s-1} n.$$

We take

$$n = \frac{1}{\delta^{s-1}} n.$$

This will also satisfy FIRST CONSTRAINT.

You got here because for all $v \in N$, for all colors c, $\deg_c^L(v) \leq \delta m$ and $\deg_c^R(v) \leq \delta m$. Hence $\deg_c(v) \leq 2\delta m$. By renumbering we assume that $N = \{1, \ldots, m\}$ and that the colors are $\{1, \ldots, m\}$. Let COL be the coloring restricted to $\binom{[m]}{2}$. Note that, for all vertices $v \in [m]$, for all colors c, $\deg_c(v) \leq 2\delta m$.

Note also that, for any vertex $v \in [m]$,

$$m - 1 < m = \sum_{c=1}^{m} \deg_c(v) \leq \sum_{c=1}^{m} \delta m = m^2 \delta.$$

Hence

THIRD CONSTRAINT:

$$\delta \geq \frac{1}{m}.$$

Note that COL is a coloring of $\binom{[m]}{2}$ such that for every v and c, $\deg_c(v) \leq 2\delta m$. Hence, by Lemma 3.2, there are at most

$$\frac{2\delta m \times m^2}{2} = \delta m^3$$

bad triples.

By Lemma 3.3 there exists a subset X of size m' that has at most

$$\delta m^3 \times \left(\frac{m'}{m}\right)^3 = \delta (m')^3$$

bad triples.

We have two options for setting m' which lead to the different upper bounds. The first option gives a simpler proof and one less parameter; however, the second option gives a better bound. We admit here that the improvement of the upper bound is marginal.

Option 1: Set m' and δ so that there are *no* bad triples. Hence we need

$$\delta(m')^3 < 1$$

We now have a set X of size m' with no bad triples. We will use Lemma 2.2 on this set, hence we take

$$m' = \frac{k^3}{2}.$$

Hence

$$\delta = \frac{2}{m'3} = \frac{16}{k^9}.$$

By THIRD CONSTRAINT we need

$$\delta \geq \frac{1}{m}.$$

We take

$$m = \frac{1}{\delta} = \frac{k^9}{16}.$$

By the SECOND CONSTRAINT

$$m \leq \delta^{s-1} n.$$

$$n = \frac{m}{\delta^{s-1}}.$$

Since $\delta = \frac{1}{m}$ we can express n in terms of m.

$$n = m^s = \left(\frac{k^9}{16}\right)^s = \left(\frac{k^9}{16}\right)^{2(k-2)^2+1}.$$

And we are DONE— with Option 1.

Option 2. We set m' such that the number of bad triples is so small that we can just remove one point from each. This will lead to a better value of n. Recall that the number of bad triples is $\delta(m')^3$.

We want the number of bad triples to be so small that if we just toss out one vertex from each we still have many (that is, m'') vertices.

FOURTH CONSTRAINT:

$$m' - \frac{\delta(m')^3}{3} \geq m''.$$

By renumbering we can assume the m'' vertices are $\{1, \ldots, m''\}$. Let COL be the coloring restricted to $\binom{[m'']}{2}$. Note that there are NO bad triples. By Lemma 2.2 there exists a rainbow set of size $(2m'')^{1/3}$. Since we want this to be $\geq k$ we have our

FIFTH CONSTRAINT:

$$m'' \geq \frac{k^3}{2}.$$

End of Phase $2c$

We now collect up all the constraints and see how to satisfy them in a way that minimizes n.

List of Constraints

1. $$\delta \geq \left(\frac{1}{n}\right)^{1/s}$$

   This constraint is implied by the next one so we do nothing.

2. $$\delta \geq \left(\frac{m}{n}\right)^{1/(s-1)}.$$

   We satisfy this by taking

   $$n = \frac{m}{\delta^{s-1}}$$

   This constraint is now satisfied; however, we need to know what m and δ are.

3. $$\delta \geq \frac{1}{m}.$$

   We will take

   $$m = \frac{1}{\delta}.$$

   This constraint is now satisfied; however, we need to know what δ is.

4.
$$m' - \frac{\delta}{3}(m')^3 \geq m''.$$

$$\delta \leq \frac{3m' - 3m''}{(m')^3}$$

Since we want δ as large as possible we will take δ to equal this upper bound. This constraint is now satisfied; however, we need to know what m', m'' are.

5.
$$m'' \geq \frac{k^3}{2}.$$

We take m'' equal to this lower bound. This constraint is now satisfied.

End of List of Constraints

$$m'' = \frac{k^3}{2}.$$

What should m' and δ be? We want to maximize δ. Recall that

$$\delta = \frac{3m' - 3m''}{(m')^3}.$$

We pick the value of $1 \leq m' \leq m$ that maximizes δ. Simple calculus reveals that this value is $m' = 1.5m''$. Hence

$$m' = 1.5m'' = \frac{1.5k^3}{2}.$$

$$\delta = \frac{3m' - 3m''}{(m')^3} = \frac{4.5m'' - 3m''}{(1.5m'')^3} = \frac{1.5m''}{(1.5m'')^3} = \frac{1}{(1.5m'')^2}$$

Note that

$$(1.5m'')^2 = \left(\frac{1.5k^3}{2}\right)^2 = \left(\frac{3k^3}{4}\right)^2 = \frac{9k^6}{16}.$$

Hence

$$\delta = \frac{16}{9k^6}.$$

Hence

$$m = \frac{1}{\delta} = \frac{9k^6}{16}.$$

We now know m and δ so we can find n. Since $m = \frac{1}{\delta}$ we express n in terms of m and then k.

$$n = \frac{m}{\delta^{s-1}} = m^s = \left(\frac{9k^6}{16}\right)^s = \left(\frac{9k^6}{16}\right)^{2(k-2)^2+1}.$$

∎

Note 3.6 Lemma 2.3 can be used to very slightly improve Theorem 3.5. We leave this to the reader.

How does the upper bound in Theorem 3.5 compare to what is known? The only bound known is $ER_3 = 4$ [3]. By contrast Theorem 3.5 yields

$$ER(3) \leq \left(\frac{9 \times 3^6}{16}\right)^3 = \frac{3^{27}}{2^{12}} \sim 1.8 \times 10^9.$$

References

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