A proof for a 4 coloring of POLYVDW($\{x^2\}$)

Srinivas Vasudevan

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The theorem we proved in class was:

Theorem 1. *Given r, $\exists N$ such that for any r coloring of $[N]$, $\exists a, d$ such that $\chi(a) = \chi(a + d^2)$*

To do so, we used Van der Waerden's theorem to prove a lemma, from which the above result follows. Another line of reasoning we used for the simple case of $r = 3$ was that $a-16, a, a+9$ must be distinct colors (assume $\chi(a) = R$) and that $\chi(a-7) = R$ because $a-7$ is a square away from $a-16, a+9$. Thus $\chi(a = k*7) = R$ so $\chi(a-7*7) = R$ a contradiction (assume N is big enough so we can do this). We more or less follow this same sort of reasoning for the case of $r = 4$. Note that: $\left| \begin{array}{c|c|c} 952^2 + 495^2 = 1073^2 & 952^2 + 561^2 = 1105^2 & 1073^2 + 264^2 = 1105^2 \\ 975^2 + 448^2 = 1073^2 & 975^2 + 520^2 = 1105^2 & 1073^2 + 264^2 = 1105^2 \end{array} \right|$

The numbers $952, 1073, 1105$ form a somewhat generalized pythagorean triple as every pairwise difference of squares is a square. Now, if we 4 color $[N]$ then $a - 952^2, a, a + 495^2, a + 561^2$ are rainbow with respect to the coloring (assume $\chi(a) = R$) because every pairwise difference is a square (using the identities above). Now if we can get $d : (a+d)-(a-952^2) = d+952^2, (a+495^2)-(a+d) = 495^2 - d, (a + 561^2) - (a + d) = 561^2 - d$ are squares, then $a - d$ must be colored R. Keeping an eye on the equations above, letting $d = 975^2 - 952^2 = 44321$ seems like a good choice.Indeed $d+952^2 = 975^2, 495^2 - d = 495^2 + 952^2 - 975^2 = 1073^2 - 975^2 = 448^2, 561^2 - d = 561^1 + 952^2 - 975^2 = 1105^2 - 975^2 = 520^2$, so $a + 44321$ must be colored R. Thus $\chi(a_k * 44321) = R$ so $\chi(a + (44321)^2) = R$ a contradiction. Thus for N sufficiently large, every 4 coloring must yield $a, d : \chi(a) = \chi(a + d^2)$. How did I pick these numbers? I searched for three numbers so that the sums and differences were squares. Note that I could have searched for three squares so that the differences were squares, however I wanted to work with more linear relations versus quadratic relations. I proved the following theorem:

Theorem 2. *There are infinitely many x, y, z, such that all pairwise sums and differences are simultaneously squares.*

After doing so, I could get generate x, y, z satisfying the theorem, which implied $x + y, y + z, x + z$ all had pairwise differences of squares (since $z - y, z - x, y - x$ are all squares). However, I generated a 1 parameter family, and I point the reader towards [1] for a proof of a 2 parameter family (for which I used. I considered integer parameterization, whereas Euler considered rational parameterization, from which he got an extra parameter). Using [1] I then set f and g for various values, to get the triple of squares $952, 1073, 1105$, and $975, 1073, 1105$. I noticed that the triples share the last 2 numbers, which allows for using $d = 975^2 - 952^2$.

References

[1] Ed Sandifer, *How Euler Did It: Sums (and differences) that are squares*, MAA, 2009, `http://www.maa.org/editorial/euler/HEDI%2065%20Sums%20that%20are%20squares.pdf`