POLY VDW THEOREM (Exposition)

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VDW

Recall:
Theorem: $(\forall k, c)(\exists W = W(k, c))$ such that for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$ such that

$$COL(a) = COL(a + d) = COL(a + 2d) = \cdots = COL(a + (k - 1)d).$$

Recall:
$W(1, c) = 1$
$W(2, c) = c + 1$ (this is PHP)
$W(k, 1) = k$

Let $W(k, c)$ mean both the number and the STATEMENT.

We proved

Recall:

$W(2,32) \implies W(3,2)$

$W(2,10^{10}) \implies W(3,3)$ (might not be big enough.)

$W(2,(10!^{10!})) \implies W(3,4)$ (might not be big enough.)

$W(3, BLAH) \implies W(4,2)$.

$W(3, BLAHBLAH) \implies W(4,3)$.

So Whats Really Going On?

Order PAIRS of naturals (think (k, c)) via

$$(2,2) \le (2,3) \le (2,4) \le \cdots \le (3,2) \le (3,3) \le (3,4) \cdots$$

$$(4,2) \le (4,3) \le \cdots (5,2) \le (5,3) \le \cdots \le (6,2) \cdots$$

Formal proof of VDW is an induction on this ordering.

Induction on an ω^2 ordering.

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(WEIRDNESS: Several HS students saw this as their FIRST proof by induction and went on to live productive lives.)

Poly VDW Theorem

Why $a, a+d, a+2d, a+3d, \ldots, a+(k-1)d$?
Replace $d, 2d, 3d, \ldots, (k-1)d$ by some other funcof d?
Is this true:
Theorem: $(\forall p_1, \ldots, p_k \in Z[x])(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$

$$COL(a) = COL(a+p_1(d)) = COL(a+p_2(d)) = \cdots = COL(a+p_k(d)).$$

VOTE!

THE REAL KEY DIFFERENCE

FALSE for a DUMB reason:

1. $k = 1$
2. $p_1(x) = 1$
3. $c = 2$.

NEED W such that for all $COL : [W] \rightarrow [2]$ there exists a, d such that

$$COL(a) = COL(a+1)$$

Take $RBRBRB \cdots$.

Poly VDW Theorem

Definition: $Z^*[x]$ are all polys with coeff in Z and zero constant term.
Theorem: $(\forall p_1, \ldots, p_k \in Z^*[x])(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$

$$COL(a) = COL(a+p_1(d)) = COL(a+p_2(d)) = \cdots = COL(a+p_k(d)).$$

RESTATE IT:

Theorem: For all finite $S \subseteq Z^*[x]$ $(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$

$$\{a\} \cup \{a + p(d) \mid p \in S\} \text{ all the same color.}$$

Notation: $PVDW(S)$ means that Poly VDW theorem holds for the set $S \subseteq Z^*[x]$. Note that

$$PVDW(x, 2x, 3x) \implies (\forall c)[VDW(3, c)].$$

Notation for Induction

Definition: A finite set $S \subseteq Z^*[x]$ is *of type* $(n_e, n_{e-1}, \ldots, n_1)$ if

- the number of diff lead coef of polys of degree e is $\le n_e$.
- the number of diff lead coef of polys of degree $e-1$ is $\le n_{e-1}$.

$\vdots$

- the number of diff lead coef of polys of degree 1 is $\le n_1$.

BILL DO EXAMPLES ON BOARD.

Better Notation

Definition: Let $(n_e, n_{e-1}, \ldots, n_1) \in \mathsf{N}^e$. $PVDW(n_e, \ldots, n_1)$ means that $PVDW(S)$ holds for all S of type $(n_e, n_{e-1}, \ldots, n_1)$.

VDW's theorem is $PVDW(1) \wedge PVDW(2) \wedge \cdots$.

We showed

$$(\bigwedge_{i \in \mathsf{N}} PVDW(i)) \implies PVDW(1,0).$$

Even Better Notation

Definition: Let $(n_e, n_{e-1}, \ldots, n_1) \in (\omega \cup \mathrm{N})^e$. $PVDW(n_e, \ldots, n_1)$ means that, for all $(m_e, \ldots, m_1) \le (n_e, \ldots, n_1)$ (component wise) $PVDW(S)$ holds for all S of type $(m_e, m_{e-1}, \ldots, m_1)$.

We showed

$$PVDW(\omega) \implies PVDW(1, 0).$$

Poly SQ VDW

Theorem: $(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$

$$COL(a) = COL(a + d^2).$$

Proof by proving Lemma:
Lemma: $(\forall c)(\forall r)(\exists U)$ for all $COL : [U] \rightarrow [c]$ EITHER

- $(\exists a, d)[COL(a) = COL(a + d^2)]$, OR
- $(\exists a, d_1, d_2, \ldots, d_r)[COL(a), COL(a + d_i^2)]$ all colored DIFFERENTLY.

BILL- REDO OR NOT IN CLASS.

PVDW(1,0)

Theorem: $(\forall c)(\forall k)(\forall A \in Z)(\forall B \subseteq Z,\ B \text{ finite})(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$
all elements of $\{a\} \cup \{a + d^2 + id : i \in B\}$ are the same color.
Proof by proving Lemma:
Lemma: $(\forall c)(\forall k)(\forall A \in Z)(\forall B \subseteq Z,\ B \text{ finite})(\exists U)$ for all $COL : [U] \rightarrow [c]$ EITHER

- $(\exists a, d)$ all elements of $\{a\} \cup \{a + d^2 + id : i \in B\}$ are the same color, OR
- $(\exists a, d_1, d_2, \ldots, d_r)$
  - $(\forall 1 \le j \le r$ the elements of $\{a + d_j^2 + id_j : i \in B\}$ are the same color. We call the jth one the jth BUBBLE.
  - All the bubbles are colored differently and all are a different color than a.

BILL- DO IN CLASS AND DO BASE CASE

PVDW(x^2, x)

Theorem: $(\forall c))(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$ all elemenents of $\{a\} \cup \{a+d, a+d^2\}$ are the same color.
Proof by proving Lemma:
Lemma: $(\forall c)(\exists U)$ for all $COL : [U] \rightarrow [c]$ EITHER

- $(\exists a, d)$ all elemenents of $\{a\} \cup \{a+d, a+d^2\}$ are the same color, OR
- $(\exists a, d_1, d_2, \ldots, d_r)$
  - ($\forall 1 \le j \le r$ the elements of $\{a+d_j, a+d_j^2\}$ are the same color. We call the jth one the jth BUBBLE.
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