Constructions in Computable Ramsey Theory (An Exposition)

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Notation

Notation:

1. $M_1, M_2, \ldots$ is a standard list of Turing Machines.
2. Note that from e we can extract the code for M_e.
3. $M_{e,s}(x)$ means that we run M_e for s steps.
4. W_e is the domain of M_e, that is,

$$W_e = \{x \mid (\exists s)[M_{e,s}(x) \downarrow].$$

   Note that $W_1, W_2, \ldots$ is a list of ALL c.e. sets.

5. $$W_{e,s} = \{x \mid M_{e,s}(x) \downarrow .$$

There is a Comp Coloring with no Inf c.e.Homog Set

Theorem

There exists computable $COL : \binom{N}{2} \to [2]$ *such that there is NO infinite c.e. homog set.*

Construction of Comp Col w/o Inf Comp Homog Set

We construct $COL : \binom{N}{2} \rightarrow [2]$ to satisfy:

$$R_e : W_e \text{ infinite } \implies W_e \text{ NOT a homog set } .$$

CONSTRUCTION OF COLORING
Stage 0: COL is not defined on anything.
Stage s: We will define $COL(0, s), COL(1, s), \ldots, COL(s-1, s)$.
For all $0 \leq e \leq s$ do the following, starting with $e = 0$:
If $(\exists x, y \leq s-1)[x, y \in W_{e,s} \wedge COL(x, s), COL(y, s)$ undefined $]$ then define take LEAST such x, y and do: (1) $COL(x, s) = RED$, (2) $COL(y, s) = BLUE$. (Note that IF $s \in W_e$ then R_e would be satisfied.)
After all this, for all (x, s) not yet colored, $COL(x, s) = RED$.
END OF CONSTRUCTION

There is a Comp Coloring with no Inf c.e.-in-HALT Homog Set

Theorem

There exists computable $COL : \binom{N}{2} \rightarrow [2]$ *such that there is NO infinite c.e-in-HALT homog set.*

This is on HW1.

Every Comp Coloring has inf Π_2 Homog Set

Theorem

For every computable coloring $COL : \binom{N}{2} \rightarrow [2]$ *there is an infinite* Π_2 *homog set.*

Construction of Inf Π_2 Homog Set

Given computable $COL : \binom{N}{2} \rightarrow [2]$.
CONSTRUCTION of $x_1, x_2, \ldots$ **and** $c_1, c_2, \ldots$.
$x_1 = x$ and $c_1 = RED$ (We are guessing. Might change later)

$s \geq 2$, assume $x_1, \ldots, x_{s-1}$ and $c_1, \ldots, c_{s-1}$ are defined.

Ask $HALT$ $((\exists x \geq x_{s-1})(\forall 1 \leq i \leq s-1)[COL(x_i, x) = c_i]$?

YES: *Find least such* x.

- $x_i = x$
- $c_i = RED$ *(Guessing.)*

Construction of Inf Π_2 Homog Set: NO Case

NO: Ask *HALT*:

- $(\exists x \ge x_{s-1})(\forall 1 \le i \le s-2)[COL(x_i, x) = c_i])$?
- $\vdots$
- $(\exists x \ge x_{s-1})(\forall 1 \le i \le 1)[COL(x_i, x) = c_i])$?

Let i_0 be largest such that
$(\exists x \ge x_{s-1})(\forall 1 \le i \le i_0)[COL(x_i, x) = c_i])$?

1. Change color of c_{i+1}.
2. Wipe out $x_{i+2}, \ldots, x_{s-1}$.
3. Find $x \ge x_{s-1}$ such that $(\forall 1 \le i \le i_0)[COL(x_i, x) = c_i]$
4. $x_{i+2} = x$. $c_{i+2} = RED$ (Guessing)

END OF CONSTRUCTION of $x_1, x_2 \ldots$ **and** $c_1, c_2, \ldots$

Getting the Inf Π_2 Homog Set

$X = \{x_1, x_2, \ldots\}$. R is the set of red elts of X
$\overline{X} \in \Sigma_2$ (so $X \in \Pi_2$).

$\overline{X} = \{x \mid (\exists s)[$ at stage s of the construction x was tossed out $]\}$.

$\overline{R} \in \Sigma_2$ (so $R \in \Pi_2$).

$\overline{R} = \overline{X} \cup \{x \mid (\exists x)[$ at stage s of the construction x was turned BLUE$]\}$.

1. If R is infinite then R is inf homog set in Π_2.
2. If R is finite then $B = X - R$ is inf homog set in Π_2.