Homework 3, Morally Due Tue Feb 20, 2018

COURSE WEBSITE: http://www.cs.umd.edu/~gasarch/858/S18.html
(The symbol before gasarch is a tilde.)

1. (0 points) What is your name? Write it clearly. Staple your HW. When is the midterm tentatively scheduled (give Date and Time)? If you cannot make it in that day/time see me ASAP.

2. (50 points) You may assume the 2-ary can ramsey theory. Recall the statement of the 3-ary Can Ramsey Theorem:

   For all ω-colorings of $\binom{\mathsf{N}}{3}$ there exists a set $I \subseteq \{1, 2, 3\}$ such that

   $$COL(x_1, x_2, x_3) = COL(y_1, y_2, y_3) \text{ iff } (\forall i \in I)[x_i = y_i].$$

   (We assume $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$.)

   Prove the 3-ary can Ramsey theory using a proof similar to Mileti's of 2-ary can Ramsey. (the one that DOES NOT use 4-ary or 3-ary hypergraph Ramsey).

   SOLUTION TO PROBLEM TWO

   We just sketch this.

   Let COL be an ω-coloring of $\binom{\mathsf{N}}{3}$.

   CONSTRUCTION PART ONE

   We construct an infinite sequence

   $X = \{x_1 < x_2 < \cdots\}$

   and an ω-coloring COL' of $\binom{X}{2}$. We will then apply The infinite 2-can Ramsey to COL'.

   The colors for COL' are (H, c) where c is a color and (R, i) where $i \in \mathsf{N}$.

   Assume that after stage $n - 1$, we have

   - $X_{n-1} = \{x_1, x_2, \ldots, x_{n-1}\}$.
   - An ω-coloring of $\binom{X_{n-1}}{2}$.
   - An infinite set V_{n-1} of numbers $> x_{n-1}$ that are still in play (have not been killed).

Let x_n be the least element of V_{n-1}.

Assume that $COL'(x_1, x_n)$, ..., $COL'(x_{i-1}, x_n)$ have been defined. We now define $COL'(x_i, x_n)$.

Case 1: If there exists c and infinitely many $y \in V_{n-1}$ such that $COL(x_i, x_n, y) = c$ then $COL'(x_i, x_n) = (H, c)$. V_{n-1} is redefined (we are not going to bother renaming it) by, you guessed it, kill all those that disagree. Formally

$$V_{n-1} = \{y : COL(x_i, x_n, y) = c\}$$

Case 2: For all c the set $\{y : COL(x_i, x_n, y) = c\}$ is finite. Hence (formally by inf 1-can Ramsey) there exists a subset of V_{n-1} where each col appears once. Formally:

$$V_{n-1} = \{y \in V_{n-1} : COL(x_i, x_n, y) \notin \{COL(x_i, x_n, z) : z \in V_{n-1} \wedge x_n < z < y\}\}$$

Case 2a: If there exists $1 \le j_1 < j_2 \le n-1$ such that

$COL'(j_1, j_2) = (R, k)$ for some k, and

$\{y \in V_{n-1} : COL(x_{j_1}, x_{j_2}, y) = COL(x_i, x_n, y)\}$ is infinite

then let

$V_n = \{y \in V_{n-1} : COL(x_{j_1}, x_{j_2}, y) = COL(x_i, x_n, y)\}$

$COL(x_i, x_n) = (R, k)$.

Case 2b: For all $1 \le j_1 < j_2 \le n-1$ such that

$COL'(j_1, j_2) = (R, k)$ for some k, and

$\{y \in V_{n-1} : COL(x_{j_1}, x_{j_2}, y) = COL(x_i, x_n, y)\}$ is finite.

By removing a finite number of vertices we can make, for all j_1, j_2 as above,

$\{y \in V_{n-1} : COL(x_{j_1}, x_{j_2}, y) = COL(x_i, x_n, y)\}$ empty.

Let k be the least number such that no edge has color (R, k).

Let $COL'(x_i, x_n) = (R, k)$.

END OF CONSTRUCTION PART ONE

SO, we have an ω-coloring of $\binom{X}{2}$. Apply the infinite 2-can Ramsey Theorem to this coloring Let A be the infinite set we get. Renumber so that $A = \mathsf{N}$.

Case 1: Suppose A is homogenous under COL'. If every edge in $\binom{A}{2}$ has color (H, c) for some fixed c, then A also homogenous under COL. Otherwise every edge has color (R, k) for some fixed k. Then if $I = \{3\}$, A is $I-$homogenous.

Case 2: Suppose that A is min-homogenous under COL'. By Ramsey's theorem, there is a subset $A' \subseteq A$ such that either all the edges in A' have type H, or all the edges have type R. In the first case, A' is $\{1\}$-homogenous. In the latter case, for all $x_1 < x_2 < x_3$ in A, $COL(x_1, x_2, x_3)$ is determined entirely by x_1 and x_3. So define $COL''(x_1, x_3)$ to be that color and let A'' be the subset of A' consisting of only the elements with odd indices. By the 2-ary canonical ramsey theorem, there is a min/max/homog/rainbow subset of A'' under COL''. Then it is also $\{1\}/\{3\}/\{\}/\{1,3\}$-homogenous set under COL.

Case 3: Suppose that A is max-homogenous under COL'. By Ramsey's theorem, there is a subset $A' \subseteq A$ such that either all the edges in A' have type H, or all the edges have type R. In the first case, A' is $\{2\}$-homogenous. In the latter case, for all $x_1 < x_2 < x_3$ in A, $COL(x_1, x_2, x_3)$ is determined entirely by x_2 and x_3. So define $COL''(x_2, x_3)$ to be that color. By the 2-ary canonical ramsey theorem, there is a min/max/homog/rainbow subset of A' under COL''. That set is also a $\{2\}/\{3\}/\{\}/\{2,3\}$-homogenous set under COL.

Case 4: Suppose that A is rainbow under COL'. By Ramsey's theorem, there is a subset $A' \subseteq$ such that either all the edges of A' are of type H or all of the edges of A' are of type R. In the first case, A' is $\{1,2\}$-homogenous. In the latter case, we can construct an infinite rainbow set. Suppose $B = \{x_1 < x_2 < \cdots < x_k\} \subseteq A'$ is a maximal rainbow set. Consider $x_k + 1$. Since B is maximal, we know that there are $x_i, x_j, x_p, x_q, x_r \in B$ such that $COL(x_i, x_j, x_k + 1) = COL(x_p, x_q, x_r)$. Since $COL'(x_i, x_j)$ is of type R, and there are only finitely many possible colors of $COL(x_p, x_q, x_r)$, the pair (x_i, x_j) can only be used in a counterexample of this form finitely many times. Since there are finitely many possible pairs (x_i, x_j), the must be some integer M such that $x_k + M$ can be safely added to B. Thus there is an infinite rainbow set.

3. (50 points) This problem is a proof technique in search of a theorem.

   Let X be a countable set of points in the plane. Color each pair by the *slope* of the line they form. Apply the Canonical Ramsey Theorem to this coloring.

   (a) Use the idea in the last paragraph to formulate a theorem.

   (b) Try to make an assumption about the points that leads to a more interesting theorem.

   SOLUTION TO PROBLEM THREE

   Theorem: For all X an infinite set of points in the plane either (1) there exists infinite $Y \subseteq X$ such that no four sets in X form a trapezoid OR (2) there exists infinite $Y \subseteq X$ such that all points in Y lie on the same line.

   Proof: Let X be a countable set of points in the plane. Color each pair by the *slope* of the line they form. Apply the Canonical Ramsey Theorem to this coloring. There are four possibilities.

   - There is an infinite homog set H. All of these points are on a line.
   - There is an infinite min-homog set H. Let the points in H be $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, ... The slope of (p_1, p_2), p_1, p_3, etc are all the same. Hence $p_1, p_2, \ldots$ are all on the same line. Hence all the points are on a line, so this is actually a homogenous set, and a min-homog set is not actually possible.
   - There is an infinite max-homog set H. Let the points in H be $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, ... Let $n \geq 3 \in \mathsf{N}$. The slope of (p_1, p_2), p_1, p_n, are the same. Hence p_1, p_2, p_n are all on the same line, and indeed all of the points in H are on the same line. So this is actually a homogenous set, and a max-homog set is not actually possible.
   - There is an infinite rainbow set H. NO two pairs of points in H have the same slope. Hence no four points form a trapezoid.

MORE INTERESTING THEOREM: The line stuff is stupid. We could assume ahead of time that there is a limit to how many points are on a line. But that seems awkward. We will assume something stronger, but a more common assumption:

Theorem: For all X an infinite set of points in the plane *in general position* (meaning that no three are colinear) there exists infinite $Y \subseteq X$ such that no four sets in X form a trapezoid.

Proof: Similar to the proof of the theorem above, except that cases 1,2,3 can't happen. **END OF SOLUTION TO PROBLEM THREE**

4. (0 points but please do as I'll ask about it in class). What did you think of the song *A finite simple group of order two* by the Klein Flour? (Link is on the website). Compare and Contrast to the BW-Rap in terms of both lyrics and singing ability.

   SOLUTION OF PROBLEM FOUR

   Lyrics: If you understand every reference you know a large cardinal amount of mathematics. They are awesome even though I don't understand them.

   Singing: They CAN sing but still they should not quit their day job. OH- they are grad students so they don't have day jobs!

   Comparison: Clearly The Klein Four are better at singing then the BW-guy is at rapping. Lyrics: Hard to compare as BW is about one theorem, while the Klein Four hit lots of parts of math.

   END OF SOLUTION OF PROBLEM FOUR