The Infinite Can Ramsey Theorem (An Exposition)

William Gasarch-U of MD

Ramsey's Theorem For Graphs

Theorem: For every $COL : \binom{\mathsf{N}}{2} \rightarrow [c]$ there is an infinite homogenous set.

What if the number of colors was infinite?

Do not necessarily get a homog set since could color EVERY edge differently. But then get infinite *rainbow set*.

Attempt

Theorem: For every $COL : \binom{\mathsf{N}}{2} \to \omega$ there is an infinite homogenous set OR an infinite rainb set.
VOTE: TRUE, FALSE, or UNKNOWN TO SCIENCE.

Attempt

Theorem: For every $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ there is an infinite homogenous set OR an infinite rainb set.

VOTE: TRUE, FALSE, or UNKNOWN TO SCIENCE.

FALSE:

- $COL(i,j) = \min\{i,j\}$.
- $COL(i,j) = \max\{i,j\}$.

Min-Homog, Max-Homog, Rainbow

Definition: Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathsf{N}$.

- V is *homogenous* if $COL(a,b) = COL(c,d)$ iff $TRUE$.
- V is *min-homogenous* if $COL(a,b) = COL(c,d)$ iff $a = c$.
- V is *max-homogenous* if $COL(a,b) = COL(c,d)$ iff $b = d$.
- V is *rainb* if $COL(a,b) = COL(c,d)$ iff $a = c$ and $b = d$.

One-Dim Can Ramsey Theorem

Lemma: Let V be an countable set. Let $COL : V \to \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).

Proof of Can Ramsey Theorem for Infinite Graphs

We are given $COL : \binom{\mathbb{N}}{2} \to \omega$.
Want to find infinite homog OR min-homog OR max-homog OR rainbow set.

We use COL to define $COL' : \binom{\mathbb{N}}{4} \to [16]$
We then apply 4-ary Ramsey theorem. (an "Application!")

In the slides below $x_1 < x_2 < x_3 < x_4$.
All cases assume negation of prior cases.

Homog always means infinite Homog.

Pairs that begin the same way are same color

1. $COL(x_1,x_2) = COL(x_1,x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 1$.
2. $COL(x_1,x_2) = COL(x_1,x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 2$.
3. $COL(x_1,x_3) = COL(x_1,x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 3$.
4. $COL(x_2,x_3) = COL(x_2,x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 4$.

H is homog set,color 1 (rest similar)
$COL'' : H \rightarrow$ N is $COL''(x) =$ color of all (x,y) with $x < y \in H$.

Use 1-dim Can Ramsey!:
Case 1: COL'' has homog set H' then H' homog for COL.
Case 2: COL'' has rainb set H' then H' min-homog for COL.

Pairs that End the same way are same color

1. $COL(x_1, x_3) = COL(x_2, x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 5.$
2. $COL(x_1, x_4) = COL(x_2, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 6.$
3. $COL(x_1, x_4) = COL(x_3, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 7.$
4. $COL(x_2, x_4) = COL(x_3, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 8.$

H is homog set,color 5 (rest similar)
$COL'' : H \rightarrow \mathrm{N}$ is $COL''(y) =$ color of all (x, y) with $x < y \in H$.

Use 1-dim Can Ramsey!:
Case 1: COL'' has homog set H' then H' homog for COL.
Case 2: COL'' has rainb set H' then H' max-homog for COL.

Easy Homog Cases

1. $COL(x_1, x_2) = COL(x_2, x_3) \Rightarrow COL(x_1, x_2, x_3, x_4) = 9.$
2. $COL(x_1, x_2) = COL(x_2, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 10.$
3. $COL(x_1, x_2) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 11.$
4. $COL(x_1, x_3) = COL(x_2, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 12.$
5. $COL(x_1, x_3) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 13.$
6. $COL(x_2, x_3) = COL(x_1, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 14.$
7. $COL(x_2, x_3) = COL(x_3, x_4) \Rightarrow COL(x_1, x_2, x_3, x_4) = 15.$

H is homog set,color 9 (rest similar)
For all $w < x < y < z \in H$.

$$COL(w, x) = COL(x, y) = COL(y, z).$$

Other cases, like $COL(w, y) = COL(x, z)$, are similar

Rainbow Case

If NONE of the above cases hold then $COL(x_1, x_2, x_3, x_4) = 16$.

Let H be homog set.

All edges from H diff colors, so Rainbow Set.

PROS and CONS of Proof

PRO: Each Case easy. Note that Rainbow case was easy.

CON: Lots of Cases. Use of 4-ary hypergraph Ramsey makes finite version have large bounds.

PROS and CONS of Proof

PRO: Each Case easy. Note that Rainbow case was easy.

CON: Lots of Cases. Use of 4-ary hypergraph Ramsey makes finite version have large bounds.
We will give anther proof which only uses 3-ary hypergraph Ramsey.

Definition that Will Help Us

Definition Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. If c is a color and $v \in \mathsf{N}$ then $\deg_c(v)$ is the number of c-colored edges with v as an endpoint.

Note: $\deg_c(v)$ could be infinite.

Needed Lemma

Lemma Let X be infinite. Let $COL : \binom{X}{2} \rightarrow \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \le 1$ then there is an infinite rainb set.
TRY TO PROVE WITH YOUR NEIGHBOR. I WILL THEN GIVE PROOF.

Proof

Let R be a MAXIMAL rainb set of X.

$$(\forall y \in X - R)[X \cup \{y\} \text{ is not a rainb set}].$$

Let $y \in X - R$. Why is $y \notin R$?

1. $(\exists u \in R, \exists\{a,b\} \in \binom{R}{2})[COL(y,u) = COL(a,b)]$.
2. $(\exists\{a,b\} \in \binom{R}{2})[COL(y,a) = COL(y,b)]$.
   If $c = COL(y,a)$ then $\deg_c(y) \geq 2$, so Can't Happen!

Map $X - R$ to $R \times \binom{R}{2}$: map $y \in X - R$ to $(u, \{a,b\})$ (item 1).
Map is injective: if y_1 and y_2 both map to $(u, \{a,b\})$ then $COL(y_1,u) = COL(y_2,u)$ but $\deg_c(u) \leq 1$.
Injection from $X - R$ to $R \times \binom{R}{2}$. If R finite then injection from an infinite set to a finite set Impossible! Hence R is infinite.

Canonical Ramsey Theorem for N

Theorem: For all $COL : \binom{\mathsf{N}}{2} \to \omega$ there is either

- an infinite homogenous set,
- an infinite min-homog set,
- an infinite max-homog set, or
- an infinite rainb set.

Proof of Can Ramsey Theorem for Graphs

Given $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. We use COL to obtain $COL' : \binom{\mathsf{N}}{3} \rightarrow [4]$
We will use the 3-ary Ramsey theorem. In all of the below $x_1 < x_2 < x_3$.

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then $COL'(x_1 < x_2 < x_3) = 1$.
2. If $COL(x_1, x_3) = COL(x_2, x_3)$ then $COL'(x_1 < x_2 < x_3) = 2$.
3. If $COL(x_1, x_2) = COL(x_2, x_3)$ then $COL'(x_1 < x_2 < x_3) = 3$.
4. If none of the above occur then $COL'(x_1 < x_2 < x_3) = 4$.

Cases 1,2,3 are just like in the prior proof.
Case 4: For all x, for all c, $\deg_c(x) \leq 1$ so have Rainbow by Lemma.

Case 4: An Alternative Proof without Maximal Sets

There is an infinite homog set of color 4: Recall: all pairs of x_1, x_2, x_3 have diff colors. Let H be the infinite homog set. Rename so

$$H = \{1, 2, 3, \ldots\}$$

GOOD NEWS: $(1, 2)$ and $(2, 3)$ diff colors.
BAD NEWS: $(1, 2)$ and $(3, 4)$ could be same color.
USEFUL NEWS: Let RE be the set of all RED edges. The set RE is a set of disjoint edges.
CANNOT have, say (4,100) and (100,200) in RE.
CANNOT have, say (4,100) and (4,200) in RE.
Need to do some more killing!

Case 4 cont:

Lets out all edges in order of max number:

$$(1,2),(1,3),(2,3),(1,4),(2,4),(3,4),(1,5),(2,5),(3,5),(4,5)\ldots$$

We process each edge.
(1,2): Say its RED. We want to KILL all RED edges but still have an infinite number of vertices. Let
(a_1, b_1), (a_2, b_2), ... be all the RED edges. KEY: all disjoint and none have 1 or 2 in them. Assume $a_i < b_i$.
KILL ALL THE b_i's!
Look at the next edge on the list thats left. Do the same.
When done have bloody rainbow set!