Ramsey's Theorem
for the Infinite Complete Graph and
the Infinite Complete Hypergraph
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1 Introduction

In this document we define notation for graphs and hypergraphs that we use for the course and then look at Ramsey's theorem and the Canonical Ramsey theory on N. Why start with N? Because Joel Spencer said

Infinite Ramsey Theory is easier than Finite Ramsey Theory because all of the messy constants go away.

2 Notation

Recall that a graph is a set of vertices and a set of edges which are *unordered pairs* of vertices. Why pairs? We will generalize this by allowing edges to be sets of size 1, 2 (the usual case), 3, general a and not have any restriction on size.

Notation 2.1

1. If $n \geq 1$ then $[n] = \{1, \ldots, n\}$.
2. If $a \in \mathsf{N}$ and A is a set then $\binom{A}{a}$ is the set of all subsets of A of size a. We commonly use $\binom{[n]}{a}$ and $\binom{\mathsf{N}}{a}$.

Def 2.2 Let $a \in \mathsf{N}$ (note that $a = 0$ is allowed). A *a-hypergraph* is a set of vertices V and a set of edges which is a subset of $\binom{V}{a}$.

Examples

1. A 0-hypergraph is just a set of vertices. This is just stupid but we'll keep it around in case we need some edge case.
2. A 1-hypergraph is a set of vertices together with edges which are also vertices. So its just a set of vertices but some are also called edges.

3. A 2-hypergraph is the usual graphs you know and love.

4. A 3-hypergraph. Edges are sets of 3 vertices. $V = \mathsf{N}$ and the edges are all (a, b, c) such that $a + b + c \equiv 0 \pmod 9$. I *could not* have said $a + 2b + 3c \equiv 0 \pmod 9$ since then the order would matter. We are dealing with unordered hypergraphs. I could have said all (a, b, c) with $a < b < c$ such that $a + 2b + 3c \equiv 0 \pmod 9$.

5. Another example of a 3-hypergraph: let V be some set of points in the plane. Let the edges be all 3-sets of points that form non-degenerate triangles.

Def 2.3 A *hypergraph* (notice the lack of a parameter) is a set of vertices V together with edges which are subsets of V.

Example

1. $V = \mathsf{N}$ and we take the set of all finite subsets of N whose sum is $\equiv 0 \pmod 9$. Note that the empty set would be an edge.

2. V is a set of points in the plane. The edges are all of the lines in the plane.

3. Any a-hypergraph is also a hypergraph.

We are all familiar with the compete graph on N:

Notation 2.4 K_{N} is the graph (V, E) where

$$\begin{aligned} V = &\ \mathsf{N} \\ E = &\ \binom{\mathsf{N}}{2} \end{aligned}$$

Here is the complete a-hypergraph on N:

Notation 2.5 K_{N}^a is the hypergraph (V, E) where

$$\begin{aligned} V = &\ \mathsf{N} \\ E = &\ \binom{\mathsf{N}}{a} \end{aligned}$$

Convention 2.6 In this course unless otherwise noted (1) a *coloring of a graph* is a coloring of the edges of the graph. and (2) a *coloring of a hypergraph* is a coloring of the edges of the hypergraph.

3 Ramsey Theory on the Complete 1-Hypergraph on N

The following theorem is to obvious to prove but I want to state it:

Theorem 3.1 *For every 2-coloring of* N *there is an infinite* $A \subseteq \mathsf{N}$ *that is the same color.*

Even though this is an easy theorem here are some questions:

1. Is there a finite version of this theorem?
2. If you are given a program that computes the coloring can you determine which color (or perhaps both) appears infinitely often?
3. What if you are given a simple computational device (e.g., a DFA with output). Then can you determine which color? What is the complexity of the problem?

What if I allow an infinite number of colors?

Theorem 3.2 *For every coloring of* N *there is either (1) an infinite* $A \subseteq \mathsf{N}$ *that is the same color, or (2) an infinite* $A \subseteq \mathsf{N}$ *that all have different colors (called a rainbow set).*

Proof: Let COL be a coloring of N. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL.

Here is the intuition: Either $COL(1)$ appears infinitely often (so we are done) or not. If not then we get rid of the finite number of vertices colored $COL(1)$ except 1. We then do the same for $COL(2)$. We will either find some color that appears infinitely often or create a sequence of all different colors.

We now describe it formally.

$$
\begin{aligned}
V_0 &= \mathsf{N} \\
x_1 &= 1 \\
\\
c_1 &= \text{DONE if } |\{v \in V_0 \mid COL(v) = COL(x_1)\}| \text{ is infinite. And you are DONE! STOP} \\
&= COL(x_1) \text{ otherwise} \\
V_1 &= \{v \in V_0 \mid COL(v) \neq c_1\} \text{ (note that } |V_1| \text{ is infinite)}
\end{aligned}
$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$
\begin{aligned}
x_i &= \text{the least number in } V_{i-1} \\
\\
c_i &= \text{DONE if } |\{v \in V_{i-1} \mid COL(v) = COL(x_i)\}| \text{ is infinite. And you are DONE! STOP} \\
&= COL(x_i) \text{ otherwise} \\
V_i &= \{v \in V_{i-1} \mid COL(v) \neq c_i\} \text{ (note that } |V_i| \text{ is infinite)}
\end{aligned}
$$

How long can this sequence go on for? If ever it stops then we are done as we have found a color appearing infinitely often. If not then the sequence

$$x_1, x_2, \ldots,$$

is infinite and each number in it is a different color, so we have found a rainbow set. ▮

1. Is there a finite version of this theorem?
2. If you are given a program that computes the coloring can you determine which color (if any) appears infinitely often?
3. What if you are given a simple computational device (e.g., a DFA with output). Then can you determine which color? What is the complexity of the problem?

4 A Bit More Notation

For the case of the 1-hypergraph we didn't need notions of complete graphs or homog sets, though that is what we were talking about. For a-hypergraphs we will.

Def 4.1 Let $COL : \binom{\mathsf{N}}{2} \rightarrow [2]$. Let $V \subseteq \mathsf{N}$. The set V is *homog* if there exists a color c such that every elements of $\binom{V}{2}$ is colored c.

Def 4.2 Let $COL : \binom{\mathsf{N}}{k} \rightarrow [c]$. Let $V \subseteq \mathsf{N}$. The set V is *homog* if there exists a color c such that every elements of $\binom{V}{k}$ is colored c.

5 Ramsey's Theorem for the Infinite Complete Graphs

The following is Ramsey's Theorem for K_{N}.

Theorem 5.1 *For every 2-coloring of the edges of K_{N} there is an infinite homog set.*

Proof: Let COL be a 2-coloring of K_{N}. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of x_1, or there are an infinite number of BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has an infinite number of edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(v, x_1) = c_1$. Then keep iterating this process.

We now describe it formally.

$$\begin{aligned}
V_0 &= \mathsf{N} \\
x_1 &= 1 \\
\\
c_1 &= \text{RED if } |\{v \in V_0 \mid COL(v, x_1) = \text{RED}\}| \text{ is infinite} \\
&= \text{BLUE otherwise} \\
V_1 &= \{v \in V_0 \mid COL(v, x_1) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}
\end{aligned}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$x_i = \text{ the least number in } V_{i-1}$$

$$\begin{aligned} c_i &= \text{RED if } |\{v \in V_{i-1} \mid COL(v, x_i) = \text{RED}\}| \text{ is infinite} \\ &= \text{BLUE otherwise} \\ V_i &= \{v \in V_{i-1} \mid COL(v, x_i) = c_i\} \text{ (note that } |V_i| \text{ is infinite)} \end{aligned}$$

(NOTE- look at the step where we define c_i. We are using the fact that if you 2-color N you are guaranteed some color appears infinitely often; we are using the 1-hypergraph Ramsey Theorem. Later when we prove Ramsey on 3-hypergraphs we will use Ramsey on 2-hypergraphs.)

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We can show by induction that, for every i, V_i is infinite. Hence the sequence

$$x_1, x_2, \ldots$$

is infinite.

Consider the infinite sequence

$$c_1, c_2, \ldots$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots$$

Denote this color by c, and consider the vertices

$$H = \{x_{i_1}, x_{i_2}, \cdots\}$$

We leave it to the reader to show that H is homog. ∎

Exercise 1 Show that, for all $c \geq 3$, for every c-coloring of the edges of K_{N}, there is a an infinite homog set.

Questions to ponder:

1. Is there a finite version?
2. What if we allow an infinite number of colors?
3. Computational and Complexity issues.

6 First "Application"

We will prove a theorem that is well known; however, this proof is by Gasarch from January 2017.

Theorem 6.1 *Let $d \in \mathsf{N}$, $d \geq 1$. If $p_1, p_2, \ldots$ is an infinite set of points in R^d. There exists a subsequence $q_1, q_2, \ldots$ such that, restricted to any coordinate, the sequence will be either strictly increasing, strictly decreasing, or constant.*

Proof: We do the proof in R^2 but all of the ideas are the same for R^d.

We define the following 9-coloring of pairs of points: Let $p_i = (x_i, y_i)$. We assume $i < j$. Then

$$COL(p_i, p_j) = \begin{cases} (DEC, DEC) & \text{if } x_i > x_j \text{ and } y_i > y_j \\ (DEC, CON) & \text{if } x_i > x_j \text{ and } y_i = y_j \\ (DEC, INC) & \text{if } x_i > x_j \text{ and } y_i < y_j \\ (CON, DEC) & \text{if } x_i = x_j \text{ and } y_i > y_j \\ (CON, CON) & \text{if } x_i = x_j \text{ and } y_i = y_j \\ (CON, INC) & \text{if } x_i = x_j \text{ and } y_i < y_j \\ (INC, DEC) & \text{if } x_i < x_j \text{ and } y_i > y_j \\ (INC, CON) & \text{if } x_i < x_j \text{ and } y_i = y_j \\ (INC, INC) & \text{if } x_i < x_j \text{ and } y_i < y_j \end{cases} \tag{1}$$

Take the homog set. Clearly it will be, in each coordinate, decreasing, constant, or increasing.

For R^d you would use the 3^d-coloring. ∎

Here is what is probably the classical proof (though I never saw the theorem until I proved it myself).

First prove the theorem for $d = 1$ then do an induction on d. The induction step is easy; however how do do the $d = 1$ case? Let $p_1, p_2, \ldots$ be a sequence of reals.

1) First Alternative Proof: Use Ramsey's theorem on pairs of numbers, the proof above but for $d = 1$. The good news- we only need to use Ramsey for 3-colors. The bad news- we are looking for non-ramsey proofs.

2) Second Alternative Proof:

There are some very easy cases whose proofs we omit and then one hard case:

1. The function $f(i) = \max\{p_1, \ldots, p_i\}$ goes to infinity. This case is easy and we leave it to you.
2. The function $f(i) = \max\{p_1, \ldots, p_i\}$ goes to negative infinity, This case is easy and we leave it to you.
3. Neither (1) nor (2) happens. Hence there exists reals $a < b$ such that $p_1, p_2, \ldots \in [a, b]$. This case we do below.

By the Bolzano-Weierstrass theorem every sequence or reals in a closed interval has a limit point (there may many limit points but we just need 1). Let p be a limit point of $p_1, p_2, \ldots$. Ther are three cases:

1. $(\forall n)(\exists i \geq n)[p_i = p]$. The sequence has a constant subsequence. Hence there is a constant subsequence and you are done.
2. $(\forall n)(\exists i \geq n)[0 < p - p_i < \frac{1}{n}]$ Hence there is an increasing subsequence and you are done.
3. $(\forall n)(\exists i \geq n)[0 < p_i - p < \frac{1}{n}]$ Hence there is an decreasing increasing subsequence and you are done.

The proof above uses Ramsey's theorem a little bit and perhaps a lot. The splitting into three cases can be regraded as using Ramsey on 1-hypergraphs. This is minor- its just the Pigeon hole principle really, and nobody in math every says *Hey! I'm using Ramsey Theory!* if they are just using that principle. More seriously- look at the proof of the BW theorem – some say it is Ramsey-like.

Also, see
`www.youtube.com/watch?v=df018klwKHget`
for a rap song about the BW theorem. Really!

7 Ramsey's Theorem for 3-Hypergraphs: First Proof

Theorem 7.1 *For all $COL : \binom{\mathsf{N}}{3} \rightarrow [2]$ there exists an infinite 3-homog set.*

Proof:
CONSTRUCTION
PART ONE

$V_0 = \mathsf{N}$

$x_0 = 1$.

Assume $x_1, \ldots, x_{i-1}$ defined, V_{i-1} defined and infinite.

$$\begin{aligned}
x_i &= \text{ the least number in } V_{i-1} \\
V_i &= V_{i-1} - \{x_i\} \text{ (Will change later without changing name.)} \\
COL^*(x,y) &= COL(x_i,x,y) \text{ for all } \{x,y\} \in \binom{V_i}{2} \\
V_i &= \text{ an infinite 2-homogeneous set for } COL^* \\
c_i &= \text{ the color of } V_i
\end{aligned}$$

KEY: for all $y, z \in V_i$, $COL(x_i, y, z) = c_i$.
END OF PART ONE

PART TWO

We have vertices

$$x_1, x_2, \ldots,$$

and associated colors

$$c_1, c_2, \ldots, .$$

There are only two colors, hence, by the 1-homog Ramsey Theorem there exists $i_1, i_2, \ldots,$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots$$

We take this color to be RED. Let

$$H = \{x_{i_1}, x_{i_2}, \ldots, \}.$$

We leave it to the reader to show that H is homog.
END OF PART TWO
END OF CONSTRUCTION
▮

Exercise 2

1. Show that, for all c, for all c-coloring of K_{N}^3 there exists an infinite 3-homog set.
2. State and prove a theorem about c-coloring $\binom{\mathsf{N}}{a}$.
3. What if we allow an infinite number of colors?

8 Ramsey's Theorem for 3-Hypergraphs: Second Proof

In the last section the proof went as follows:

- Color a *node* by using 2-hypergraph Ramsey. This is done ω times.
- After the nodes are colored we use 1-hypergraph. This is done once.

We given an alternative proof where:

- Color an *edge* by using 1-hypergraph Ramsey This is done ω times.
- After *all* the edges of an infinite complete graph are colored we use 2-hypergraph Ramsey. This is done once.

We now proceed formally.

Theorem 8.1 *For all* $COL : \binom{\mathsf{N}}{3} \rightarrow [2]$ *there exists an infinite 3-homog set.*

Proof:
Let COL be a 2-coloring of $\binom{\mathsf{N}}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots,$$

Here is the intuition: Let $x_1 = 1$ and $x_2 = 2$. The vertices x_1, x_2 induces the following coloring of $\{3, 4, \ldots\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let V_1 be an infinite 1-homogeneous for COL^*. Let $COL^{**}(x_1, x_2)$ be the color of V_1. Let x_3 be the least vertex left (bigger than x_2).

The number x_3 induces *two* colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[COL_1^*(y) = COL(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[COL_2^*(y) = COL(x_2, x_3, y)]$$

Let V_2 be an infinite 1-homogeneous for COL_1^*. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Restrict COL_2^* to elements of V_2, though still call it COL_2^*. We reuse the variable name V_2 to be an infinite 1-homogeneous for COL_2^*. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Let x_4 be the least element of V_2. Repeat the process.

We describe the construction formally.

CONSTRUCTION
PART ONE:

$$\begin{aligned} x_1 = & \ 1 \\ V_1 = & \ \mathsf{N} - \{x_1\} \end{aligned}$$

Let $i \geq 2$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and $COL^{**} : \binom{\{x_1,\ldots,x_{i-1}\}}{2} \rightarrow \{\text{RED}, \text{BLUE}\}$ are defined.

$$\begin{aligned} x_i = & \ \text{the least element of } V_{i-1} \\ V_i = & \ V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name).} \end{aligned}$$

We define $COL^{**}(x_1, x_i)$, $COL^{**}(x_2, x_i)$, $\ldots$, $COL^{**}(x_{i-1}, x_i)$. We will also define smaller and smaller sets V_i (not smaller by size – they are all infinite – but smaller by being subsets). We will keep the variable name V_i throughout.

For $j = 1$ to $i - 1$

1. $COL_j^* : V_i \rightarrow \{\text{RED}, \text{BLUE}\}$ is defined by $COL_j^*(y) = COL(x_j, x_i, y)$.
2. Let V_i be redefined as an infinite 1-homogeneous set for COL^*. Note that V_i is still infinite.
3. $COL^{**}(x_j, x_i)$ is the color of V_i.

END OF PART ONE

PART TWO:

From PART ONE we have a set of vertices X

$$X = \{x_1, x_2, \ldots\}$$

and a 2-coloring COL^{**} of $\binom{X}{2}$. By the 2-hypergraph Ramsey Theorem there exists an infinite homog (with respect to COL^{**}) set

$$H = \{y_1, y_2, \ldots\}$$

Assume that the homog color is R. Then for $i < j < k$

$$COL(y_i, y_j, y_k) = COL^{**}(y_i, y_j) = R$$

So H is homog for COL. **END OF PART TWO**
END OF CONSTRUCTION ▮