Take Home Midterm. Given out Feb 27
Morally Due THURSDAY March 8. Sick Cat Day-TUESDAY March 13
FIVE PAGES!!!!!!!!!!!!!!!

1. (0 points) What is your name? Write it clearly. Staple this.

2. (25 points) Find a function $f(n)$ such that the following is true, and prove it:

   - For any coloring (any number of colors) of $\{1, \ldots, f(n)\}$ there exists either n elements that are the same color OR there exists n elements that are all different colors.
   - There exists a coloring (any number of colors) of $\{1, \ldots, f(n)-1\}$ with neither n elements that are the same color NOR with n elements that are all different colors.

3. (25 points)

   (a) Find a function $f(n)$ such that the following is true, and prove it using a maximal-set argument.

   If X is a set of points in the plane, no three colinear, of size $f(n)$ then there exists $Y \subseteq X$ of size n such that no four points form a trapezoid.

   (b) Find a function $f(n,k)$ such that the following is true, and prove it using a maximal-set argument. (We assume $n, k \geq 3$.)

   If X is a set of points in the plane, no k colinear, of size $f(n,k)$ then there exists $Y \subseteq X$ of size n such that no four points form a trapezoid.

GO TO NEXT PAGE

4. (25 points) Let COL be a coloring of $\mathsf{N} \times \mathsf{N}$. A *mono grid* is a pair of sets $A, B \subseteq \mathsf{N}$ such that the COL restricted to $A \times B$ is monochromatic. If A and B are both of size infinite we say its an *infinite mono grid of size n.* If A and B are both of size n we say its an *mono grid of size n.*

   (a) Prove or disprove: For all 2-colorings of $\mathsf{N} \times \mathsf{N}$ there exists an infinite mono grid.

   (b) Find a function $f(n)$ such that the following is true (and prove it), or show that no such function exists:

   For all 2-colorings of $[f(n)] \times [f(n)]$ *there exists a mono grid of size* n.

   (c) Find a function $f(n, c)$ such that the following is true (and prove it), or show that no such function exists:

   For all c-colorings of $[f(n, c)] \times [f(n, c)]$ *there exists a mono grid of size* n.

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5. (50 points) In this problem we guide you through a finite version of Mileti's proof of the infinite can Ramsey Theorem. We work backwards by taking the last part of the proof first.

ADVICE: (1) When the infinite proof asked for an INFINITE subset, here instead take a subset that is of size square root of what we had, (2) make gross overestimates to get this all to work – trying to refine it gets complicated.

PROBLEM MILLONE

Find a function $f(n)$ such that the following lemma holds.

Lemma Let COL be an ω-coloring of $\binom{[f(n)]}{2}$. Assume that

- For all $1 \le i \le f(n) - 2$, for all $i < k_1 < k_2 \le f(n)$

$$COL(i, k_1) \ne COL(i, k_2).$$

- For all $1 \le i < j \le f(n) - 1$, for all $k \ge j + 1$,

$$COL(i, k) \ne COL(j, k).$$

Then there exists a rainbow set of size n. (Note that we DO NOT have one yet since $COL(3, 8) = COL(4, 11)$ is possible.)

PROBLEM MILLTWO Find a function $g(n)$ such that the following lemma is true: **Lemma** Let COL' be a coloring of $[g(n)]$ where the colors are of the form (H, c) and (RB, i). Then one of the following must occur:

(a) There exists c and $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored (H, c).

(b) There exists $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored $(H, *)$ and they all have different second components.

(c) There exists i and $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored (RB, i).

(d) There exists $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored $(RB, *)$ and they all have different second components.

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PROBLEM MILLTHREE

Find a function $h(n)$ such that the following lemma is true: **Lemma** Let COL be an ω-coloring of $\binom{[h(n)]}{2}$ Assume there is a coloring COL' of $[h(n)]$ where the colors are of the form (H,c) and (RB,i), and the following holds:

- If $COL'(x) = (H,c)$ then for all $z > x$ $COL(x,z) = c$.
- If $COL'(x) = (RB,i)$ then for all $z_1 \neq z_2 > x$, $COL(x,z_1) \neq COL(x,z_2)$.
- If $COL'(x) = (RB,i)$ and $COL'(y) = (RB,i)$ then for all $z > \max\{x,y\}$, $COL(x,z) = COL(y,z)$.
- If $COL'(x) = (RB,i)$ and $COL'(y) = (RB,j)$ (with $i \neq j$) then for all $z > \max\{x,y\}$, $COL(x,z) \neq COL(y,z)$.

Then one of the followings holds:

(a) There is a homog set of size n.

(b) There is a min-homog set of size n.

(c) There is a max-homog set of size n.

(d) There is a rainbow set of size n.

PROBLEM MILLFOUR

Find a function $BILL(n)$ (sorry, I'm running out of letters) such that the following lemma is true: **Lemma:** Let COL be a ω-coloring of $\binom{[BILL(n)]}{2}$ Then there is a subset of $[BILL(n)]$ of size n and a coloring COL' of that subset, where the colors are of the form (H,c) and (RB,i), such that the following holds:

- If $COL'(x) = (H,c)$, then for all $z > x$, $COL(x,z) = c$.
- If $COL'(x) = (RB,i)$, then for all $z_1, z_2 > x$, $COL(x,z_1) \neq COL(x,z_2)$.
- If $COL'(x) = (RB,i)$ and $COL'(y) = (RB,i)$, then for all $z > \max\{x,y\}$, $COL(x,z) = COL(y,z)$.
- If $COL'(x) = (RB,i)$ and $COL'(y) = (RB,j)$ (with $i \neq j$), then for all $z > \max\{x,y\}$, $COL(x,z) \neq COL(y,z)$.

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PROBLEM MILLFIVE Put all of this together to (easily) find a function $CR(n)$ (for Can Ramsey) such that the following theorem is true:

Theorem Let COL be an ω-coloring of $\binom{[CR(n)]}{2}$. Then one of the following holds:

(a) There is a homog set of size n.

(b) There is a min-homog set of size n.

(c) There is a max-homog set of size n.

(d) There is a rainbow set of size n.

6. (25 points) (This is a NEW problem – nothing to do with Finite Can Ramsey.) Let $(L, \preceq)$ be a well quasi order. Let $2^{\mathrm{fin}L}$ be the set of FINITE subsets of L. We DEFINE an order $\preceq'$ on $2^{\mathrm{fin}L}$:

$A \preceq' B$ if there is an injection f from A to B such that $x \preceq f(x)$.

($\emptyset \preceq' B$ is always true: use the empty function and the condition holds vacuously.)

Show that $(2^{\mathrm{fin}L}, \preceq')$ is a well quasi order.

(NOTE- this proof will use that wqo are closed under cross product, but the proof I have does not use Ramsey Theory directly.)