1 Grid Colorings Extension is Hard

Notation 1.1 If $x \in \mathbb{N}$ then $[x]$ denotes the set $\{1, \ldots, x\}$. $G_{n,m}$ is the set $[n] \times [m]$.

Notation 1.2 If X is a set and $k \in \mathbb{N}$ then $\binom{X}{k}$ is the set of all size-k subsets of X.

On November 30, 2009 the following challenge was posted on Complexity Blog [2].

BEGIN EXCERPT

The 17×17 challenge: worth \$289.00. I am not kidding.

Def 1.3 A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a,b),(a+c_1,b),(a+c_1,b+c_2),(a,b+c_2)\}$ for some $a,b,c_1,c_2 \in \mathbb{N}$. Note that we are only looking at the four corners of the rectangle, nothing else. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi : G_{n,m} \to [c]$ such that there are no rectangles with all four corners the same color. In other words, a grid $G_{n,m}$ is *c-colorable* if there is a function $\chi : G_{n,m} \to [c]$ such that there are no monochromatic rectangles.

The 17×17 **challenge:** The first person to email me a 4-coloring of $G_{17,17}$ in LaTeX will win \$289.00. (289.00 is chosen since it is 17^2.)
END EXCERPT

There are two motivations for this kind of problem. (1) The problem of coloring grids to avoid rectangles is a relaxation of the classic theorem (a corollary of the Gallai-Witt theorem) which states that for a large enough grid any coloring yields a monochromatic square, and (2) grid-coloring problems avoiding rectangles are equivalent to finding certain bipartite Ramsey Numbers. For more details on these motivations, and why the four-coloring of $G_{17,17}$ was of particular interest, see the post [2] or the paper by Fenner, et al [1].

Brian Hayes, the Mathematics columnist for Scientific American, publicized the challenge [5]. Initially there was a lot of activity on the problem. Some used SAT solvers, some used linear programming, and one person offered an exchange: *buy me a \$5000 computer and I'll solve it*. Finally in 2012 Bernd Steinbach and Christian Posthoff [8, 4] solved the problem. They used a rather clever algorithm with a SAT solver. They believed that the solution was close to the limits of their techniques.

Though this particular instance of the problem was solved, the problem of grid coloring in general seems to be difficult. In this paper we formalize and prove three different results that indicate grid coloring is hard.

1.1 Grid Coloring Extension is NP-Complete

Between the problem being posed and resolved the following challenge was posted [3] though with no cash prize. We paraphrase the post.

BEGIN PARAPHRASE

Def 1.4 Let $c, N, M \in \mathbb{N}$.

1. A mapping χ of $G_{N,M}$ to $[c]$ is a *c-coloring* if there are no monochromatic rectangles.

2. A partial mapping χ of $G_{N,M}$ to $[c]$ is *extendable to a c-coloring* if there is an extension of χ to a total mapping which is a c-coloring of $G_{N,M}$. We will use the term *extendable* if the c is understood.

Def 1.5 Let
$$\text{GCE} = \{(N, M, c, \chi) : \chi \text{ is extendable}\}.$$
GCE stands for *Grid Coloring Extension*.

CHALLENGE: Prove that GCE is NP-complete.
END PARAPHRASE

We show that GCE is indeed NP-complete. This result may explain why the original 17×17 challenge was so difficult. Then again—it may not. GCE with parameter c is Fixed Parameter Tractable.

There is another reason the results obtained may not be the reason why the 17×17 challenge was hard. The 17×17 challenge can be rephrased as proving that $(17, 17, 4, \chi) \in$ GCE where χ is the empty partial coloring. This is a rather special case of GCE since *none* of the spots are pre-colored. It is possible that GCE in the special case where χ is the empty coloring is easy. While we doubt this is true, we note that we have not eliminated the possibility.

One could ask about the problem

$$GC = \{(n, m, c) : G_{n,m} \text{ is } c\text{-colorable }\}.$$

However, this does not quite work. If n, m are in unary, then GC is a sparse set. By Mahaney's Theorem [7, 6] if a sparse set is NP-complete then P = NP. If n, m are in binary, then we cannot show that GC is in NP since the obvious witness is exponential in the input. This formulation does not get at the heart of the problem, since we believe it is hard because the number of possible colorings is large, not because n, m are large. It is an open problem to find a framework within which a problem like GC can be shown to be hard.

2 GCE **is NP-complete**

Theorem 2.1 GCE *is NP-complete.*

Proof:

Clearly GCE $\in$ NP.

Let $\phi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m$ be a 3-CNF formula. We determine N, M, c and a partial c-coloring χ of $G_{N,M}$ such that

$$(N, M, c, \chi) \in \text{GCE iff } \phi \in \text{3-SAT.}$$

The grid will be thought of as a main grid with entries to the left and below the main rid which are only there to enforce that some of the colors in the main grid occur only once. The colors will be T, F, and some of the $(i, j) \in G_{N,M}$. We use (i, j) to denote a color for a particular position.

The construction is in four parts. We summarize the four parts here before going into details.

1. We will often need to define $\chi(i, j)$ to be (i, j) and then never have the color (i, j) appear in any other cell of the main grid. By doing this we make having a monochromatic rectangle rare and have control over when that happens.

   We show how to color the cells that are not in the main grid to achieve this. While we show this first, it is actually the last step of the construction.

2. The main grid will have $2nm + 1$ rows. In the first column we have $2nm$ blank spaces and the space $(1, 2nm + 1)$ colored with $(1, 2nm + 1)$. The $2nm$ blank spaces will be forced to be colored T or F. We think of the column as being in n blocks of $2m$ spaces each. In the ith block the coloring will be forced as follows:

   (a) If x_i is to be set to F then the column is colored:

   $$\begin{matrix} T \\ F \\ \vdots \\ T \\ F \end{matrix}$$

   (b) if x_i is to be set to F then the column is colored:

   $$\begin{matrix} F \\ T \\ \vdots \\ F \\ T \end{matrix}$$

3. For each clause C there will be two columns. The coloring χ will be defined on most of the cells in these columns. However, the coloring will extend to these two columns iff one of the literals in C is colored T in the first column.

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4. We set the number of colors properly so that the T and F will be forced to be used in all blank spaces.

1) Forcing a color to appear only once in the main grid.

Say we want the cell $(2,4)$ in the main grid to be colored $(2,4)$ and we do not want this color appearing anywhere else in the main grid. We can do the following: add a column of $(2,4)$'s to the left end (with one exception) and a row of $(2,4)$'s below. Here is what we get:

$(2,4)$								
$(2,4)$								
T		$(2,4)$						
$(2,4)$								
$(2,4)$								
$(2,4)$								
$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$

(The double lines are not part of the construction. They are there to separate the main grid from the rest.)

It is easy to see that in any coloring of the above grid the only cells that can have the color $(2,4)$ are those shown to already have that color. It is also easy to see that the color T we have will not help to create any monochromatic rectangles since there are no other T's in its column. The T we are using *is* the same T that will later mean TRUE. We could have used F. We do not want to use new colors since we would have no control over where else they could be used.

What if some other cell needs to have a unique color? Lets say we also want to color cell $(5,3)$ in the main grid with $(5,3)$ and do not want to color anything else in the main grid $(5,3)$. Then we do the following:

$(5,3)$	$(2,4)$								
$(5,3)$	$(2,4)$								
$(5,3)$	T		$(2,4)$						
T	$(2,4)$					$(5,3)$			
$(5,3)$	$(2,4)$								
$(5,3)$	$(2,4)$								
$(5,3)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$	$(2,4)$
$(5,3)$	$(5,3)$	$(5,3)$	$(5,3)$	$(5,3)$	$(5,3)$	$(5,3)$	$(5,3)$	$(5,3)$	$(5,3)$

It is easy to see that in any coloring of the above grid the only cells that can have the color $(2,4)$ or $(5,3)$ are those shown to already have those colors.

For the rest of the construction we will only show the main grid. If we denote a color as D (short for *Distinct*) in the cell (i,j) then this means that (1) cell (i,j) is color (i,j) and (2) we have used the above gadget to make sure that (i,j) does not occur as a color in any other cell of the main grid. Note that we when we have D in the $(2,4)$ cell and in the $(5,3)$ cell they denote different colors.

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2) **Forcing $(x, \overline{x})$ to be colored (T, F) or (F, T).**

There will be one column with cells labeled by literals. The cells are blank, uncolored. We will call this row *the literal column*. We will put to the left of the literal column, separated by a triple line, the literals whose values we intend to set. These literals are not part of the construction; they are a visual aid. The color of the literal-labeled cells will be T or F. We need to make sure that all of the x_i have the same color and that the color is different than that of $\overline{x}_i$.

Here is an example which shows how we can force $(x_1, \overline{x}_1)$ to be colored (T, F) or (F, T).

$\overline{x}_1$		T	F
x_1		T	F

We will actually need m copies of x_1 and m copies of $\overline{x}_1$. We will also put a row of D's on top which we will use later. We illustrate how to do this in the case of $m = 3$.

	D	D	D	D	D	D	D	D	D	D	D
$\overline{x}_1$		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
$\overline{x}_1$		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
$\overline{x}_1$		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

We leave it as an exercise to prove that

- If the bottom x_1 cell is colored T then (1) all of the x_1 cells are colored T, and (2) all of the $\overline{x}_1$ cells are colored F.

- If the bottom x_1 cell is colored F then (1) all of the x_1 cells are colored F, and (2) all of the $\overline{x}_1$ cells are colored T.

Note that (1) if we want one literal-pair (that is $x_1, \overline{x}_1$) then we use two columns, (2) if we want two literal-pairs then we use six columns, and (3) if we want three literal-pairs then we use ten columns. We leave it as an exercise to generalize the construction to m literal-pairs using $2 + 4(m - 1)$ columns.

We will need m copies of x_2 and m copies of $\overline{x}_2$. We illustrate how to do this in the case of $m = 2$. We use double lines in the picture to clarify that the x_1 and the x_2 variables are not chained together in any way.

	D	D	D	D	D	D	D	D	D	D	D	D	D
$\overline{x}_2$		D	D	D	D	D	D	D	D	D	D	T	F
x_2		D	D	D	D	D	D	D	D	T	F	T	F
$\overline{x}_2$		D	D	D	D	D	D	T	F	T	F	D	D
x_2		D	D	D	D	D	D	T	F	D	D	D	D
$\overline{x}_1$		D	D	D	D	T	F	D	D	D	D	D	D
x_1		D	D	T	F	T	F	D	D	D	D	D	D
$\overline{x}_1$		T	F	T	F	D	D	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D	D	D

We leave it as an exercise to prove that, for all $i \in \{1, 2\}$:

- If the bottom x_i cell is colored T then (1) all of the x_i cells are colored T, and (2) all of the $\overline{x}_1$ cells are colored F.

- If the bottom x_i cell is colored F then (1) all of the x_i cells are colored F, and (2) all of the $\overline{x}_1$ cells are colored T.

An easy exercise for the reader is to generalize the above to a construction with n variables with m literal-pairs for each variable. This will take $n(2 + 4(m - 1))$ columns.

For the rest of the construction we will only show the literal column and the clause columns (which we define in the next part). It will be assumed that the D's and T's and F's are in place to ensure that all of the x_i cells are one of $\{T, F\}$ and the $\overline{x}_i$ cells are the other color.

3) How we force the coloring to satisfy ONE clause

Say one of the clauses is $C_1 = L_1 \vee L_2 \vee L_3$ where $L_1, L_2,$ and L_3 are literals. Pick an L_1 row, an L_2 row, and an L_3 row. We will also use the top row, as we will see. For other clauses you will pick other rows. Since there are m copies of each variable and its negation this is easy to do.

The two T's in the top row in the next picture are actually in the very top row of the grid.

We put a C_1 over the columns that will enforce that C_1 is satisfied. We put L_1, L_2, and L_3 on the side to indicate the positions of the variables. These C_1 and the L_i outside the triple bars are not part of the grid. They are a visual aid.

		C_1	C_1
	D	T	T
L_3		D	F
L_2			
L_1		F	D

Claim 1: If χ' is a 2-coloring of the blank spots in this grid (with colors T and F) then it CANNOT have the L_1, L_2, L_3 spots all colored F.

Proof of Claim 1:

Assume, by way of contradiction, that that L_1, L_2, L_3 are all colored F. Then this is what it looks like:

		C_1	C_1
	D	T	T
L_3	F	D	F
L_2	F		
L_1	F	F	D

The two blank spaces are both FORCED to be T since otherwise you get a monochromatic rectangle of color F. Hence we have

		C_1	C_1
	D	T	T
L_3	F	D	F
L_2	F	T	T
L_1	F	F	D

This coloring has a monochromatic rectangle which is colored T. This contradicts χ' being a 2-coloring of the blank spots.

End of Proof of Claim 1

We leave the proof of Claim 2 below to the reader.

Claim 2: If χ' colors L_1, L_2, L_3 anything except F, F, F then χ' can be extended to a coloring of the grid shown.

Upshot: A 2-coloring of the grid is equivalent to a satisfying assignment of the clause.

Note that each clause will require 2 columns to deal with. So there will be $2m$ columns for this.

4) **Putting it all together**

Recall that $\phi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m$ is a 3-CNF formula.

We first define the main grid and later define the entire grid and N, M, c.

The main grid will have $2nm + 1$ rows and $n(4m - 2) + 2m + 1$ columns. The first $n(4m - 2) + 1$ columns are partially colored using the construction in Part 2. This will establish the literal column. We will later set the number of colors so that the literal column must use the colors T and F.

For each of the m clauses we pick a set of its literals from the literals column. These sets-of-literals are all disjoint. We can do this since we have m copies of each literal-pair. We then do the construction in Part 3. Note that this uses two columns. Assuming that all of the D's are colored distinctly and that the only colors left are T and F, this will ensure that the main grid is c-colorable iff the formula is satisfiable.

The main grid is now complete. For every (i, j) that is colored (i, j) we perform the method in Part 1 to make sure that (i, j) is the only cell with color (i, j). Let the number of such (i, j) be C. The number of colors c is $C + 2$. ∎

3   An Example

We can make the construction slightly more efficient (and thus can actually work out an example). We took m pairs $\{x_i, \overline{x}_i\}$. We don't really need all m. If x_i appears in a clauses and $\overline{x}_i$ appears in b clauses then we only need $\max\{a,b\}$ literal-pairs. If $a \neq b$ then we only need $\max\{a,b\} - 1$ literal-pairs and one additional literal. (This will be the case in the example below.)

With this in mind we will do an example- though we will only show the main grid.

$$(x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

We only need

- one $(x_1, \overline{x}_1)$ literal-pair,

- one $(x_2, \overline{x}_2)$ literal-pair,

- one $(x_3, \overline{x}_3)$ literal-pair,

- one additional $\overline{x}_3$,

- one $(x_4, \overline{x}_4)$ literal-pair.

												C_1	C_1	C_2	C_2	C_3	C_3
	D	D	D	D	D	D	D	D	D	D	D	T	T	T	T	T	T
$\overline{x}_4$		D	D	D	D	D	D	D	D	T	F	D	D	D	D	D	F
x_4		D	D	D	D	D	D	D	D	T	F	D	D	D	F	D	D
$\overline{x}_3$		D	D	D	D	D	D	T	F	D	D	D	D	D	D	D	D
x_3		D	D	D	D	T	F	T	F	D	D	D	D			D	D
$\overline{x}_3$		D	D	D	D	T	F	D	D	D	D	D	F	D	D		
$\overline{x}_2$		D	D	T	F	D	D	D	D	D	D	D	F	D	D	D	
x_2		D	D	T	F	D	D	D	D	D	D			D	D	D	D
$\overline{x}_1$	T	F	D	D	D	D	D	D	D	D		D	D	D	D	F	D
x_1	T	F	D	D	D	D	D	D	D	D		F	D	D	D	D	D

4   Fixed Parameter Tractability

The 17×17 problem only involved 4-colorability. Does the result that GCE is NP-complete really shed light on the hardness of the 17×17 problem? What happens if the number of colors is fixed?

Def 4.1 Let $c \in \mathbb{N}$. Let

$$\text{GCE}_c = \{(N, M, \chi) : \chi \text{ can be extended to a } c\text{-coloring of } G_{N,M} \ \}.$$

Clearly $GCE_c \in DTIME(c^{O(NM)})$. Can we do better? Yes. We will show that GCE is in time $O(N^2 M^2 + 2^{O(c^4)})$.

Lemma 4.2 *Let n, m, c be such that $c \leq 2^{nm}$. Let χ be a partial c-coloring of $G_{n,m}$. Let U be the uncolored grid points. Let $|U| = u$. There is an algorithm that will determine if χ can be extended to a full c-coloring that runs in time $O(cnm2^{2u}) = 2^{O(nm)}$.*

Proof: For $S \subseteq U$ and $1 \leq i \leq c$ let

$$f(S, i) = \begin{cases} YES & \text{if } \chi \text{ can be extended to color } S \text{ using only colors } \{1, \ldots, i\}; \\ NO & \text{if not.} \end{cases} \tag{1}$$

We assume throughout that the coloring χ has already been applied.

We are interested in $f(U, c)$; however, we use a dynamic program to compute $f(S, i)$ for all $S \subseteq U$ and $1 \leq i \leq c$. Note that $f(\emptyset, i) = YES$.

We describe how to compute $f(S, i)$. Assume that for all S' such that $|S'| < |S|$, for all $1 \leq i \leq c$, $f(S', i)$ is known.

1. For all nonempty 1-colorable $T \subseteq S$ do the following (Note that there are at most 2^u sets T.)

   (a) If $f(S - T, i) = NO$ then $f(S, i) = NO$.

   (b) If $f(S - T, i - 1) = YES$ then determine if coloring T with i will create a monochromatic rectangle. If not then $f(S, i) = YES$. Note that this takes $O(nm)$.

2. We now know that for all 1-colorable $T \subseteq S$ (1) $f(S - T, i) = YES$, and (2) either $f(S - T, i - 1) = NO$ or $f(S - T, i - 1) = YES$ and coloring T with i creates a monochromatic rectangle. We will show that in this case $f(S, i) = NO$.

Assume that, for all 1-colorable sets $T \subseteq S$: (1) $f(S - T, i) = YES$, and (2) either $f(S - T, i - 1) = NO$ or $f(S - T, i - 1) = YES$ and coloring T with i creates a rectangle with χ. Also assume, by way of contradiction, that $f(S, i) = YES$. Let COL be an extension of χ to S. Let T be the set colored i. Clearly $f(S - T, i - 1) = YES$. Hence the second clause of condition (2) must hold. Hence coloring T with i creates a monochromatic rectangle. This contradicts COL being a c-coloring.

The dynamic program fills in a table that is indexed by the 2^u subsets of S and the c colors. Each slot in the table takes $O(nm2^u)$ to compute. Hence filling the entire table takes $O(cnm2^{2u})$ steps. ∎

Lemma 4.3 *Assume $c + 1 \leq N$ and $c\binom{c+1}{2} < M$. Then $G_{N,M}$ is not c-colorable. Hence, for any χ, $(N, M, \chi) \notin GCE_c$.*

Proof: Assume, by way of contradiction, that there is a c-coloring of $G_{N,M}$. Since every column has at least $c+1$ elements the following mapping is well defined: Map every column to the least $(\{i,j\}, a)$ such that the $\{i,j\} \in \binom{[c+1]}{2}$ and both the ith and the jth row of that column are colored a. The range of this function has $c\binom{c+1}{2}$ elements. Hence some element of the range is mapped to at least twice. This yields a monochromatic rectangle.   ▌

Lemma 4.4 *Assume $N \leq c$ and $M \in \mathbb{N}$. If χ is a partial c-coloring of $G_{N,M}$ then $(N, M, \chi) \in \mathrm{GCE}_c$.*

Proof: The partial c-coloring χ can be extended to a full c-coloring as follows: for each column use a different color for each blank spot, making sure that all of the new colors in that column are different from each other.   ▌

Theorem 4.5 $\mathrm{GCE}_c \in DTIME(N^2 M^2 + 2^{O(c^6)})$ *time.*

Proof:

1. Input (N, M, χ).

2. If $N \leq c$ or $M \leq c$ then test if χ is a partial c-coloring of $G_{N,M}$. If so then output YES. If not then output NO. (This works by Lemma 4.4.) This takes time $O(N^2 M^2)$. Henceforth we assume $c + 1 \leq N, M$.

3. If $c\binom{c+1}{2} < M$ or $c\binom{c+1}{2} < N$ then output NO and stop. (This works by Lemma 4.3.)

4. The only case left is $c + 1 \leq N, M \leq c\binom{c+1}{2}$. By Lemma 4.2 we can determine if χ can be extended in time $O(2^{NM}) = O(2^{c^6})$.

Step 2 takes $O(N^2 M^2)$ and Step 4 takes time $2^{O(c^6)}$), hence the entire algorithm takes time $O(N^2 M^2 + 2^{O(c^6)})$.   ▌

Can we do better? Yes, but it will require a result from [1].

Lemma 4.6 *Let $1 \leq c' \leq c - 1$.*

1. *If $N \geq c + c'$ and $M > \frac{c}{c'}\binom{c+c'}{2}$ then $G_{N,M}$ is not c-colorable.*

2. *If $N \geq 2c$ and $M > 2\binom{2c}{2}$ then $G_{N,M}$ is not c-colorable. (This follows from a weak version of the $c' = c - 1$ case of Part 1.)*

Theorem 4.7 $\mathrm{GCE}_c \in DTIME(N^2 M^2 + 2^{O(c^4)})$ *time.*

Proof:

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1. Input (N, M, χ).

2. If $N \le c$ or $M \le c$ then test if χ is a partial c-coloring of $G_{N,M}$. If so then output YES. If not then output NO. (This works by Lemma 4.4.) This takes time $O(N^2 M^2)$.

3. For $1 \le c' \le c - 1$ we have the following pairs of cases.

    (a) $N = c + c'$ and $M > \frac{c}{c'}\binom{c+c'}{2}$ then output NO and stop. (This works by Lemma 4.6.)

    (b) $N = c + c'$ and $M \le \frac{c}{c'}\binom{c+c'}{2}$. By Lemma 4.2 we can determine if χ can be extended to a total c-coloring in time $2^{O(NM)}$. Note that $MN \le (c + c')\frac{c}{c'}\binom{c+c'}{2}$. On the interval $1 \le c' \le c - 1$ this function achieves its maximum when $c' = 1$. Hence this case takes $2^{O(c^4)}$.

    Henceforth we assume $2c \le N, M$.

4. If $M > 2\binom{2c}{2}$ or $N > 2\binom{2c}{2}$ then output NO and stop. (This works by Lemma 4.6.)

5. The only case left is $2c \le N, M \le 2\binom{2c}{2}$. By Lemma 4.2 we can determine if χ can be extended in time $2^{O(NM)} \le 2^{O(c^4)}$.

Step 2 and Step 4 together take time $O(N^2 M^2 + 2^{O(c^4)})$. ∎

Even for small c the additive term $2^{O(c^4)}$ is the real timesink. A cleverer algorithm that reduces this term is desirable. By Theorem 2.1 this term cannot be made polynomial unless P=NP.

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