**Planar Three Dimensional Matching is NP-Complete (An Exposition)**

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# 1 Introduction

Dyer and Frieze [1] showed that the planar three dimensional matching problem (henceforth Planr 3DM) is NP-complete. We give an exposition of their proof.

We define 3DM and Planar 3DM.

**Problem 1.1** *3DM and Planar 3DM.*

*INSTANCE: 3 disjoint sets $R, B, Y$ with equal cardinality, $q$ in each set, and a set $T$ made of triples from $R \times B \times Y$ (i.e. $(\forall t \in T)[t \in R \times B \times Y]$). An instance can be represented as a bipartite graph: $R \cup B \cup Y$ on the left, $T$ on the right, and $x \in R \cup B \cup Y$ is conntected to $t \in T$ iff $x \in t$. The problem is Planar 3DM if this graph is planar.*

*QUESTION: Determine whether there is a subset of $q$ triples which contain all of the elements of $R \cup B \cup Y$.*

To prove Planar 3DM is NP-completene, we will do the following:

1. Recall that Planar 3-SAT is known to be NP-complete.

2. Give a reduction from Planar 3-SAT to Planar 1-3SAT. Hence Planar 1-3SAT is NP-complete.

3. Give a reduction from Planar 1-3SAT to Planar X3C. Hence Planar X3C is NP-complete.

4. Give a reduction from Planar X3C to Planar 3DM. Hence Planar 3DM is NP-complete.

# 2 Planar 1-3SAT is NP-Complete

**Problem 2.1** *Planar 1-3SAT.*

*INSTANCE: A Planar 3CNF formula $\phi$.*

*QUESTION: Is there a satisfying assignment where every clause has exactly one literal set to TRUE?*

**Theorem 2.2** *Planar 1-3SAT is* NP*-complete.*

**Proof:** We show Planar 3SAT $\le_p$ Planar 1-3SAT.

1. Input a planar 3CNF formula $\phi$. We can assume each claue has exactly 3 literals.
2. For each clause $C = L_1 \vee L_2 \vee L_3$ we do the following. First note that the graph of $\phi$ is Figure 1, (Left). Replace this part of the graph with the clauses and variables represented in Figure 1, (Right).
3. The resulting formula is $\phi'$.

We leave it to the reader to show that $\phi'$ is planar and that $\phi$ is in Planar 3SAT iff $\phi'$ is in Planar 1-3SAT.

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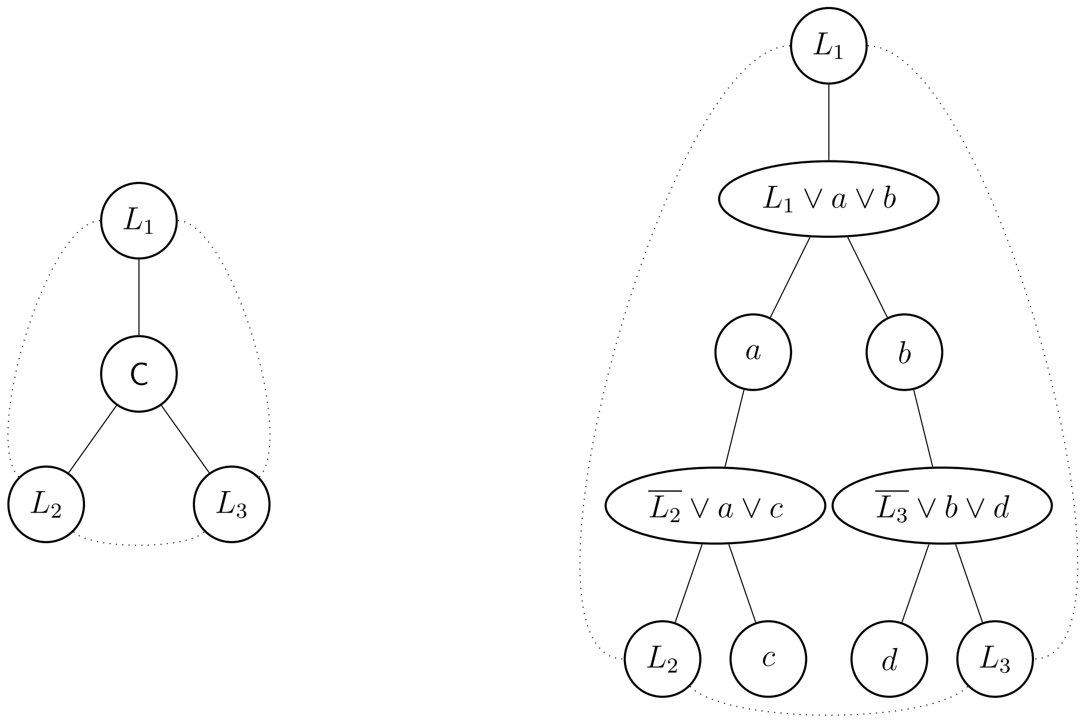



Figure 1: Planar 3SAT $\leq$ Planar 1-3SAT

# 3 Planar Exact Covering by 3-Sets (X3C)

**Problem 3.1** *X3C and Planar X3C.*

*INSTANCE:* $n \equiv 0 \pmod 3$ *and sets* $E_1, \ldots, E_m \subseteq \{0, \ldots, n\}$ *where each* $E_i$ *is of size 3. An instance can be represented as a bipartite graph:* $\{0, \ldots, n\}$ *on the left, and* $E_1, \ldots, E_m$ *on the right.* $x \in \{0, \ldots, n\}$ *conntected to* $E_i$ *iff* $x \in E_i$. *The problem is Planar X3C if this graph is planar.*

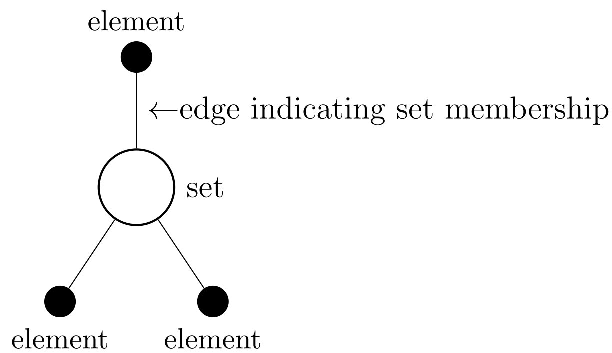
*QUESTION: Do some* $\frac{n}{3}$ *of the* $E_i$*'s cover all of* $\{0, \ldots, n\}$*, without overlapping?*

**Theorem 3.2** *Planar X3C is* NP*-complete.*

**Proof:**

We show Planar 1-3SAT $\leq_p$ Planar X3C.

Before continuing we point out how we will view the sets $E$. Figure 3 shows how: we will have a big white circle representing $E$ and then edges to black circles that represent the elements of $E$.




And now we give the reduction. When we say $x$ *occurs in* $r$ *clauses* we mean that there are $r$ clauses that have $x$ or $\neg x$.

1. Input is a planar 3-CNF formula $\phi$.

2. For each variable $x$ in $\phi$ do the following: Form a cycle of sets. If $x$ occurs $r$ times in the instance, then the cycle has $2r$ sets with each pair of sets sharing an element. For the case of $r = 3$, see Figure 2. Let the sets around the cycle in the Figure be labelled, in order starting from the left most, $1, 2, 3, 4, 5, 6$ $(1, \ldots, 2r$ in the general case). Note that to cover all of the elements of the cycle one can either take $1, 3, 5$ or $2, 4, 6$. These will correspond to setting $x$ to $T$ or $F$. Note that in each case only 3 of the 6 external elements are covered ($r$ of $2r$ in the general case). Let the clauses where $x$ or $\neg x$ appears be $C_{i_1}, C_{i_2}, C_{i_3}$. where $i_1 < i_2 < i_3$. We will associate $C_{i_1}$ with external nodes 1 and 2, $C_{i_2}$ with external nodes 3 and 4, and $C_{i_3}$ with external nodes 5 and 6 (we leave it to you to write down the general case).

3. For each clause $C$ in $\phi$ do the following. Let $x$ be a variable in $C$. Note that we already have a cycle build for $x$ and two external nodes associated to $C$. Note that the two external nodes are connected as follows:

   BILL TO ED AND JACOB: PUT IN THE FIGURE THAT IS JUST THE BASE OF THE CONNECTOR FIGURE.

   We then put the gadget in Figure 3 on top of this line of 5 nodes.

   BILL TO ED AND JACOB: DO WE USE ONE OF THE CONNECTORS IF $x$ IS IN $c$ and THE OTHER ONE IF $\neg x$ IS IN $C$?

See Figures 3 and 4 below for this illustration.

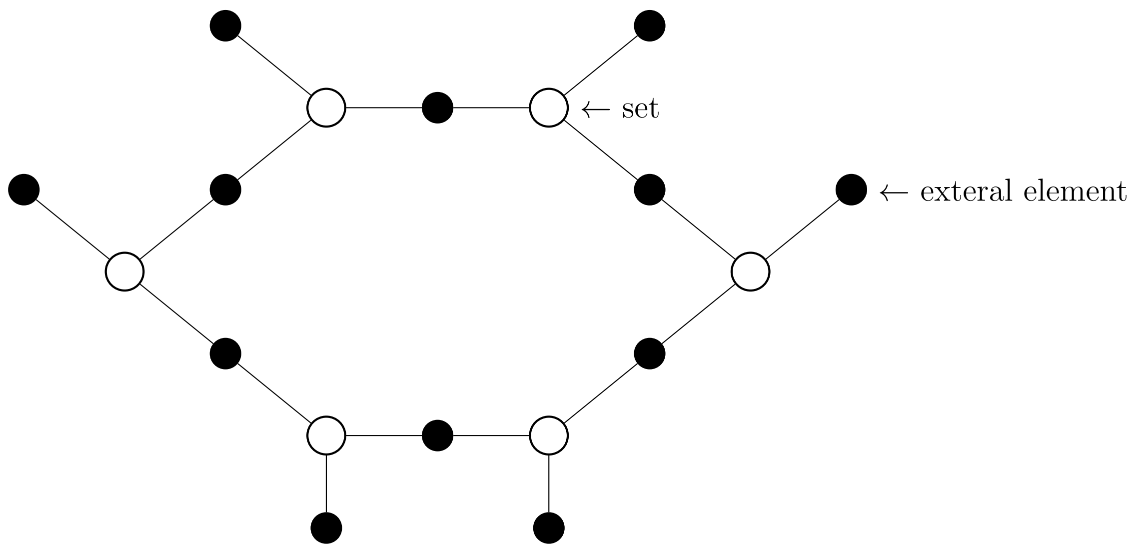



Figure 2: The Cycle Representing $x$ in the $r = 3$ case

Augment the cycle with $r$ additional sets and $2r$ elements by adding a 3-set to one of the external elements in each pair.
See Figures 3 and 4 below for this illustration.

BILL TO AUTHORS: YOU NEED TO SAY MORE CLEARLY WHAT YOU DO WITH CLAUSES.

BILL TO AUTHORS: ALL OF THE EXTERNAL ELEMENTS SHOULD BE LABELLED AS SUCH.

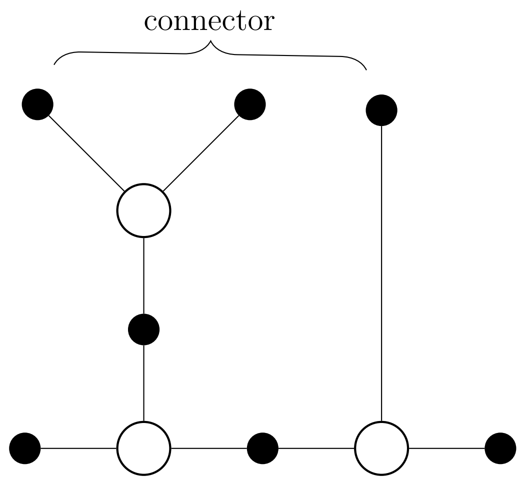



Figure 3: All three connector elements are covered

The three elements of $b$ will be denoted in the figures as a connector. Either all three, or none of the connected elements will be covered by the sets of the augmented $b$ cycle.

We must verify if negation is handled correctly. Figure 5 below represents a clause in the 1-3SAT instance. A group of 3 external elements is called a *terminal*:

To complete the construction of an X3C instance, identify the three connector elements for $b$ in $L_1$ with one of the terminals of $L_1$.

Now, we have to verify that there is an exact cover of this $L_1$ configuration.

In this configuration, the 3 internal elements each appear in 3 of the 9 sets.

Thus, 3 of the sets are used and 9 of the 12 elements will be covered internally, and one terminal will be left uncovered.

By using symmetry in Figure 5, we can verify that, if a terminal is covered externally, the remaining elements will be covered internally. Thus, there is

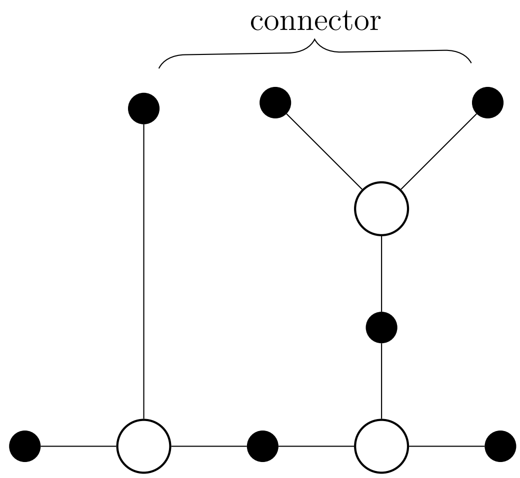



Figure 4: None of the connector elements are covered

an exact cover by 3-sets for this *planar* X3C instance iff there is a satisfying truth assignment for the planar 1-3SAT instance.

This establishes the NP-completeness of Planar X3C.

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# 4 Planar 3DM

**Theorem 4.1** *Planar 3DM is* NP*-complete.*

**Proof:**

We will prove this using a reduction Planar X3C $\leq_p$ Planar 3DM.

We will do this by modifying the X3C instance to show that the elements can be colored red (R), blue (B), or yellow (Y), such that each 3-set is incident with one element of each color.

The cycles in Figure 2 have a coloring such that:

(i) All external elements are B (colored blue)

(ii) Internal elements are alternately colored R and Y

This is shown below in Figure 6

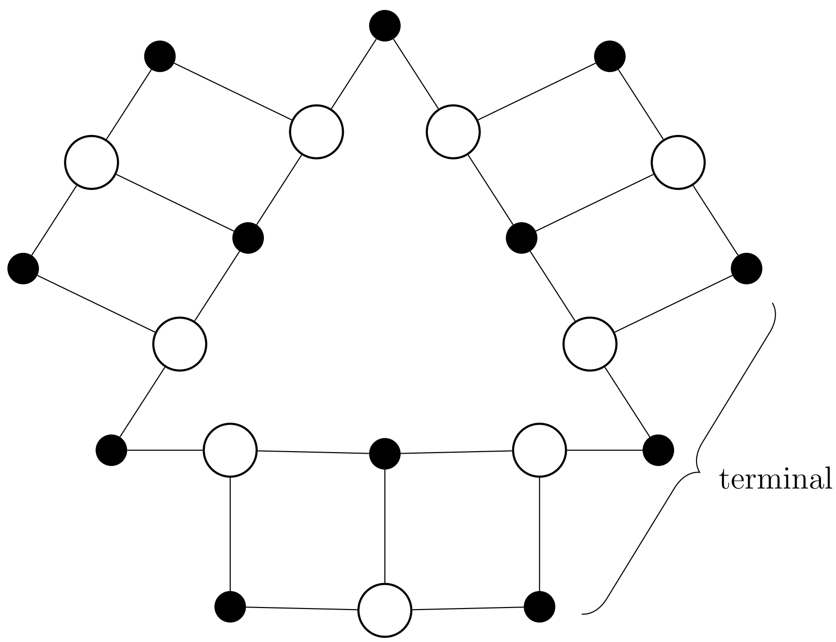


Figure 5: A graphical representation of a clause, for example $L_1$, in the 1-3SAT instance. There are 9 sets, and 12 elements.

The connector elements can be colored so that the 3 elements are colored differently.

The B element is the fixed connector element, but R, Y elements have freedom regarding which of {R, Y} they are colored.

The clause in Figure 5 has a 3-coloring in which the three terminals have coloring, from left to right,

(i) RBY

(ii) BYR

(iii) YRB

An example of scenario (i) is illustrated Figure 7.

The three internal elements each receive a different coloring.

In order to match the connector elements with the terminals, we would need to augment the variable cycles if the fixed connector element needs to

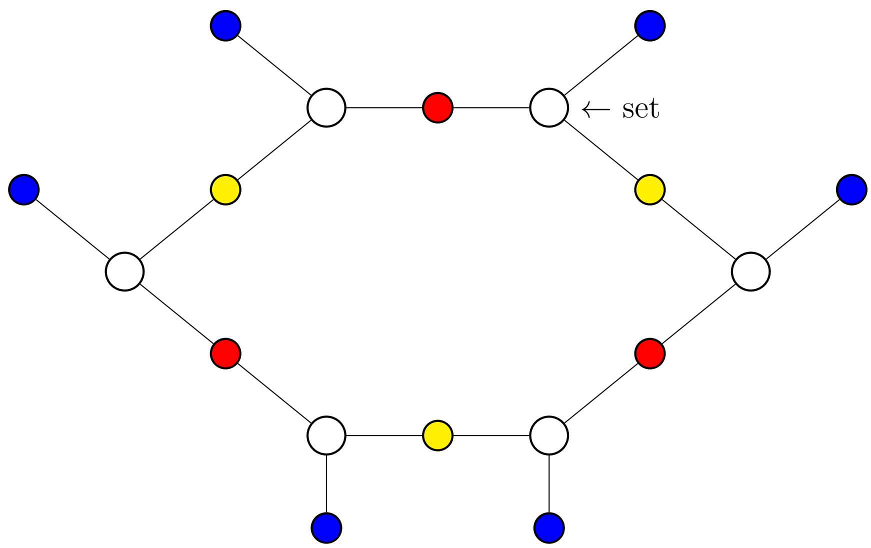


Figure 6: with colorings

be colored R or Y. See Figure 8 for this configuration.
Using this component, we can match all terminals by changing the coloring if necessary.

This establishes that Planar 3DM is NP-completeness of Planar 3DM.

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# References

[1] M. E. Dyer and A. M. Frieze. Planar 3dm is NP-complete. *J. Algorithms*, 7(2):174–184, 1986. `https://doi.org/10.1016/0196-6774(86)90002-7`.

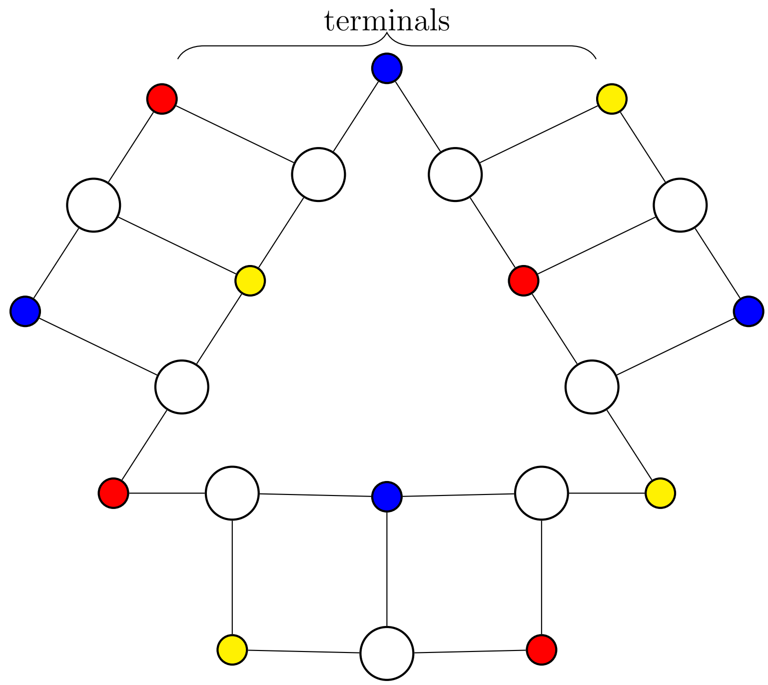


Figure 7: Colored example of Figure 5

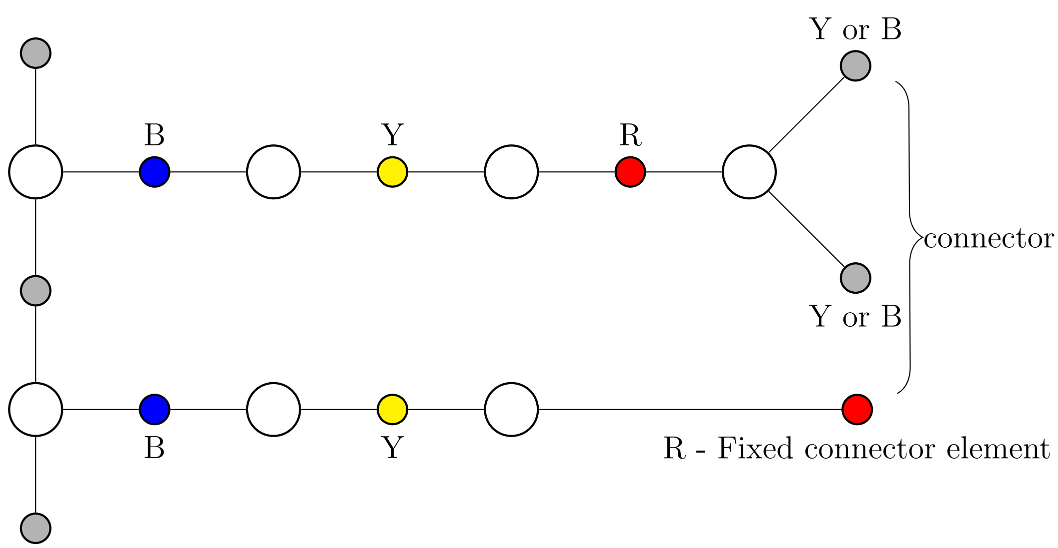


Figure 8: The augmented $b$ cycle.