Sicherman Dice

William Gasarch - University of MD

Different Labels on Dice

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Can You Label Dice To Get Same Probs?

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We prove this on the next slide.

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1. 12: TWO ways. Prob $\frac{1}{18}$.
2. 9: THREE ways. Prob $\frac{1}{12}$.
3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.

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4. 6: TWO ways. Prob $\frac{1}{18}$.
5. 5: TWELVE ways. Prob $\frac{1}{3}$.
6. 4: FOUR ways. Prob $\frac{1}{9}$.
7. 3: THREE ways. Prob $\frac{1}{12}$.
8. 2: ONE ways. Prob $\frac{1}{36}$.

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$$(x^6+x^5+x^4+x^3+x^2+x)(x^6+x^5+x^4+x^3+x^2+x).$$

Is there a Non-Standard Labeling That... Cont.

$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)^2 = x^2(x^5+x^4+x^3+x^2+x+1) =$$

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

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So desired dice are $(1,2,2,3,3,4)$ and $(1,3,4,5,6,8)$.

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Answer on Next Slide

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The proof is similar to what we did, though requires some thought.

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   Or maybe just **Unknown to Bill**.

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2. Polynomials are useful for problems with dice since multiplication gives information.
3. It is remarkable that a problem about dice lead to looking at complex roots of polynomials!