

BILL, RECORD LECTURE!!!!

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Monochromatic C_4

Exposition by William Gasarch

Credit Where Credit is Due

The main theorem in these slides was proven by

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Here is a link

<https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/c4.pdf>

When Do You Get A Mono C_4 ?

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- 1) $R(C_4) = 18$.
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Answer on the next page.

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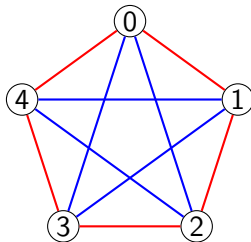
We present a COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono C_4 .

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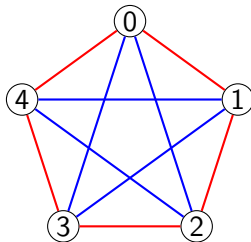


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We know that there is a mono triangle.

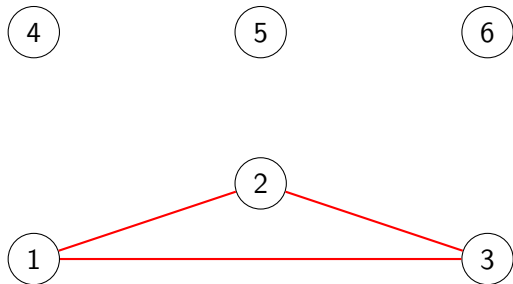
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$\exists$ a mono triangle. We assume **R** and on vertices $\{1, 2, 3\}$.

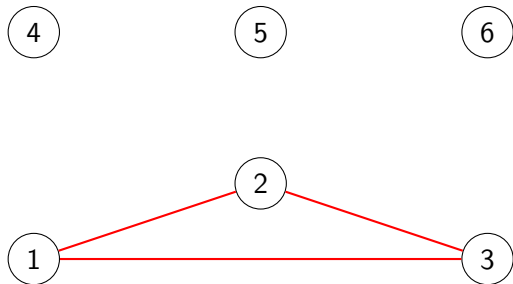
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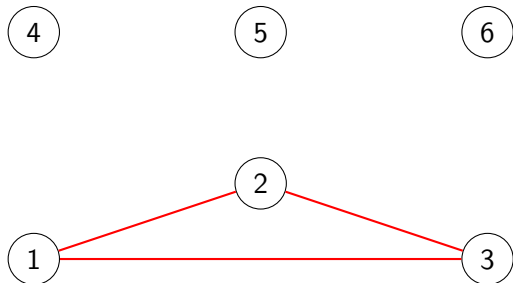
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We view $\{1, 2, 3\}$ and $\{4, 5, 6\}$ as the sides of a bipartite graph.

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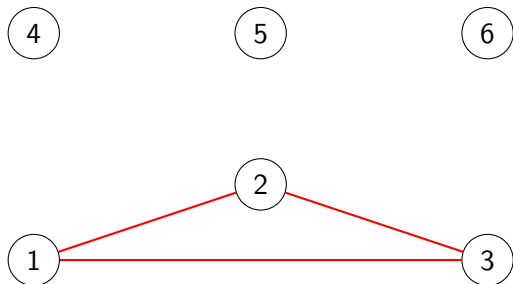
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 $\deg_{\mathbf{R}}(1)$ will mean the number of **R** edges between 1 and $\{4, 5, 6\}$.

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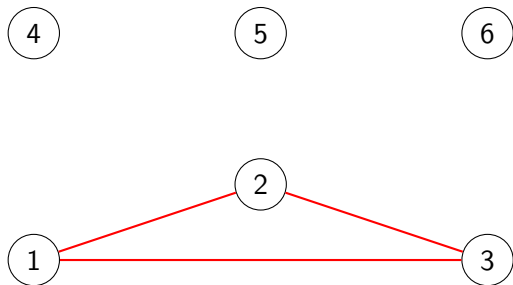
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$\deg_{\mathbf{R}}(4)$ will mean the number of **R** edges between 4 and $\{1, 2, 3\}$.

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Generalize to $\deg_{\mathbf{R}}(v)$ and $\deg_{\mathbf{B}}(v)$.

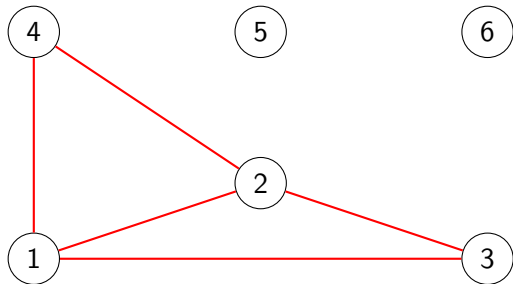
R Degree of 4, 5, 6

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If $\exists v \in \{4, 5, 6\}$, $\deg_R(v) \geq 2$ then get C_4 :

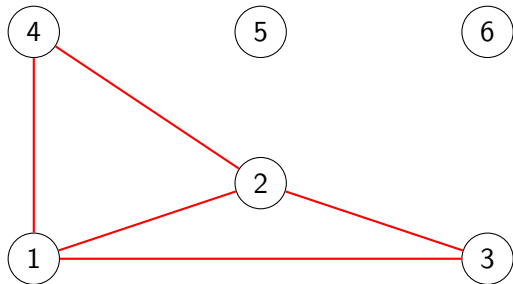
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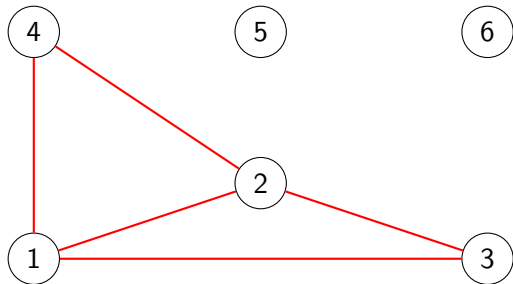
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C_4 : $4 - 1 - 3 - 2 - 4$.

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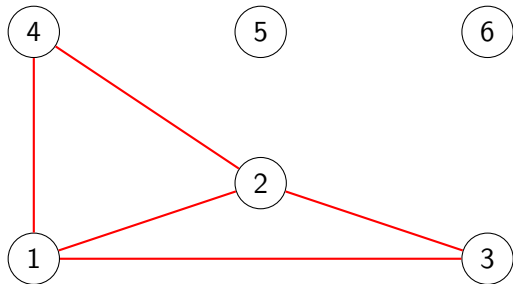


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Note: $(\forall v \in \{4, 5, 6\})[\deg_B(v) \geq 2]$.

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We will show that for $(\forall v \in \{4, 5, 6\})[\deg_B(v) = 2]$.

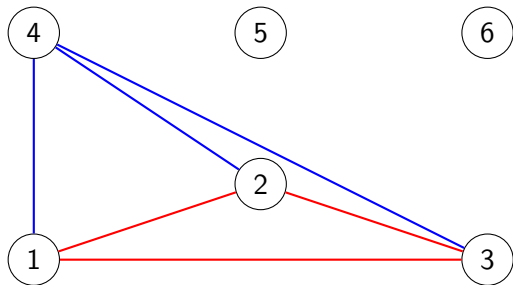
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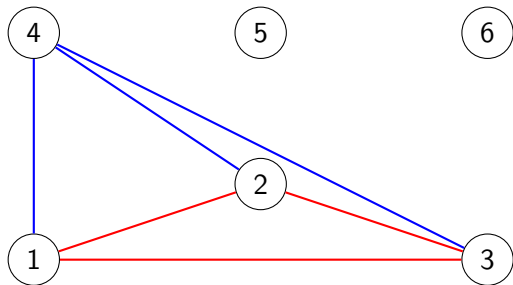
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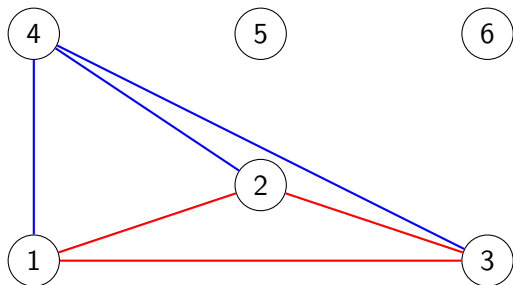
If $(\exists v \in \{4, 5, 6\})[\deg_B(v) = 3]$ then get C_4 :



Recall that $\deg_B(5) \geq 2$.

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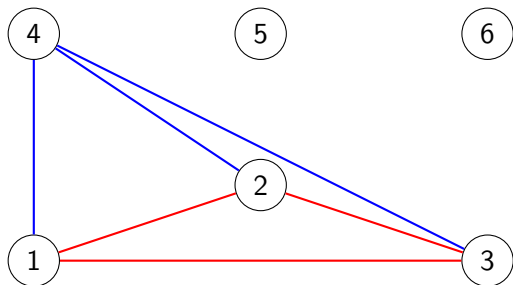


Recall that $\deg_B(5) \geq 2$.

If $\text{COL}(5, 1) = \text{COL}(5, 2) = \mathbf{B}$ then C_4 : $5 - 1 - 4 - 2 - 5$.

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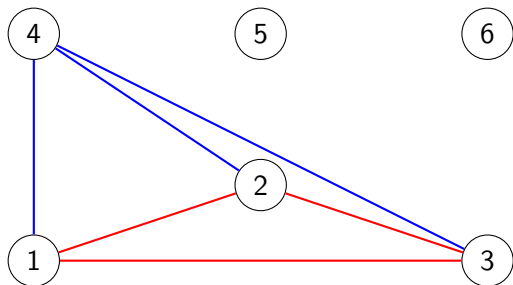
Recall that $\deg_B(5) \geq 2$.

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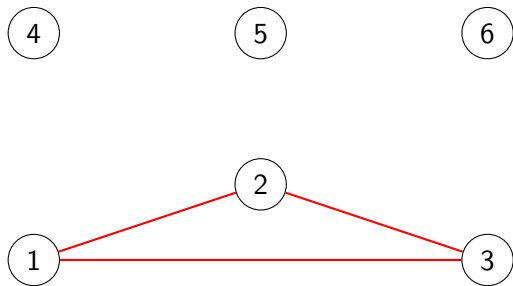
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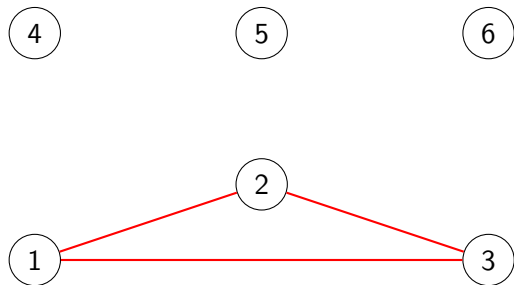
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Recap

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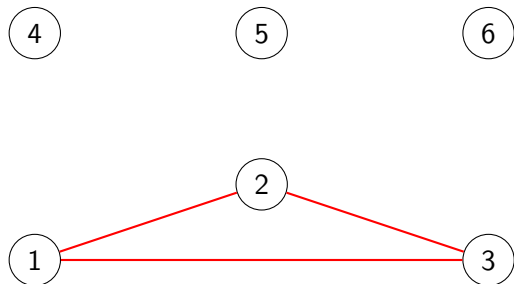


Recap



For $v \in \{4, 5, 6\}$, $\deg_{\mathbf{R}}(v) = 1$ and $\deg_{\mathbf{B}}(v) = 2$.

Recap



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We now look at the degrees of 1, 2, 3.

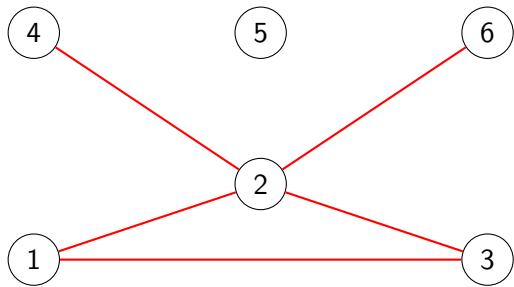
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If $\exists v \in \{1, 2, 3\}$ $\deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C}_4$ or $\mathbf{C}_4$.

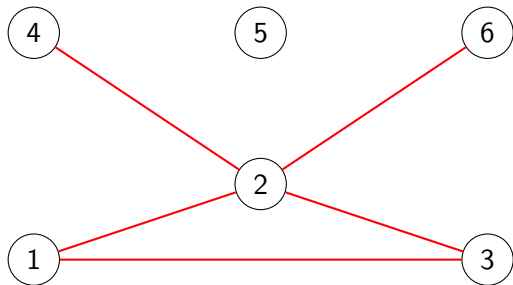
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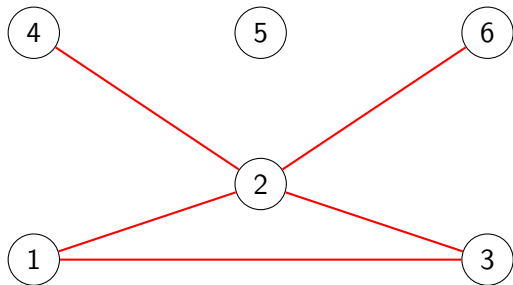
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If $\text{COL}(1, 4) = \mathbf{R}$ then $\mathbf{C}_4$: $1 - 4 - 2 - 3 - 1$.

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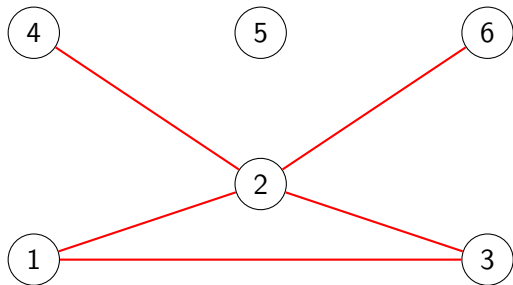


If $\text{COL}(1, 4) = \mathbf{R}$ then $\mathbf{C}_4$: $1 - 4 - 2 - 3 - 1$.

If $\text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4$: $3 - 6 - 2 - 1 - 3$.

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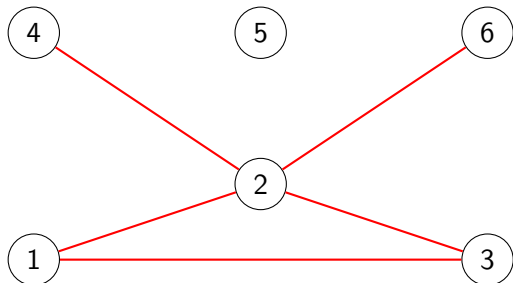
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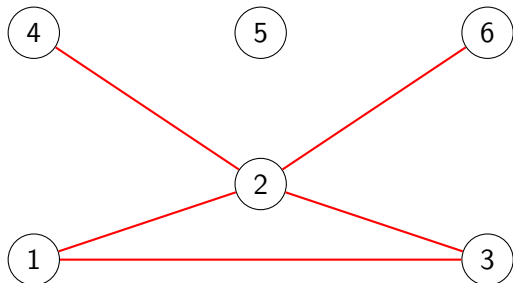
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For $v \in \{1, 2, 3\}$, $\deg_{\mathbf{R}}(v) = 1$

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If $\text{COL}(1, 6) = \mathbf{R}$ then $\mathbf{C}_4$: $1 - 6 - 2 - 3 - 1$.

If all of those edges are $\mathbf{B}$ then $\mathbf{C}_4$: $1 - 4 - 3 - 6 - 1$.

Red Degree Recap

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► $(\forall v \in \{1, 2, 3\})[\deg_{\mathbf{R}}(v) = 1].$

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- ▶ $(\forall v \in \{4, 5, 6\})[\deg_{\mathbf{R}}(v) = 1]$.

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- ▶ $(\forall v \in \{4, 5, 6\})[\deg_{\mathbf{R}}(v) = 1]$.
- ▶ Hence we can assume

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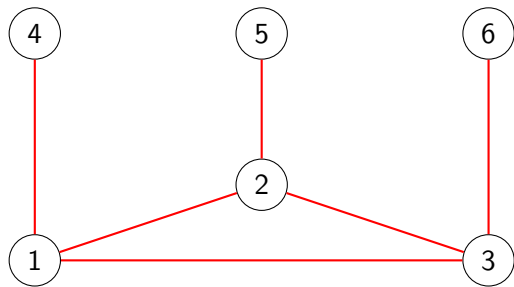
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- ▶ Hence we can assume
 - ▶ $\text{COL}(1, 4) = \text{COL}(2, 5) = \text{COL}(4, 6) = \mathbf{R}$

Red Degree Recap

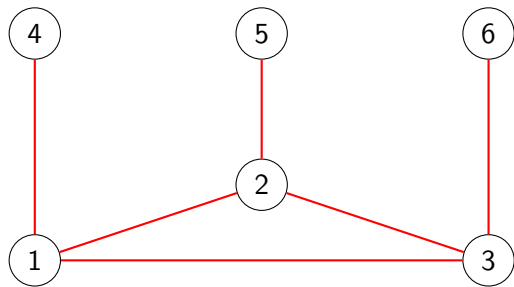
We have

- ▶ $(\forall v \in \{1, 2, 3\})[\deg_{\mathbf{R}}(v) = 1]$.
- ▶ $(\forall v \in \{4, 5, 6\})[\deg_{\mathbf{R}}(v) = 1]$.
- ▶ Hence we can assume
 - ▶ $\text{COL}(1, 4) = \text{COL}(2, 5) = \text{COL}(4, 6) = \mathbf{R}$
 - ▶ All other edges between $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are $\mathbf{B}$. (We will find some other edges that must be $\mathbf{B}$.)

What We Know: R

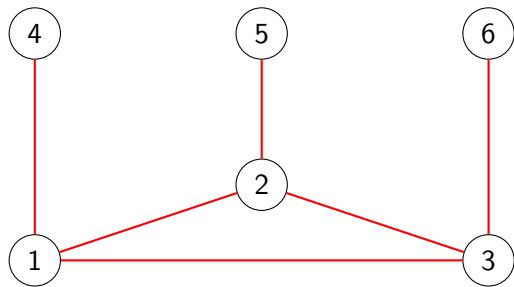


What We Know: **R**



All edges between $\{1, 2, 3\}$ and $\{4, 5, 6\}$ not shown are **B**.

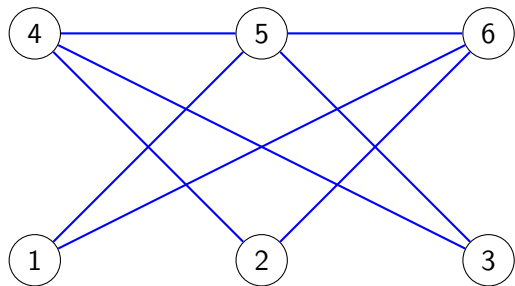
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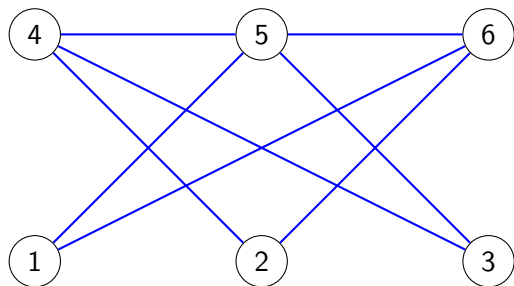
All edges between $\{1, 2, 3\}$ and $\{4, 5, 6\}$ not shown are **B**.

Clearly $\text{COL}(4, 5) = \text{COL}(5, 6) = \mathbf{B}$.

What We Know: **B**

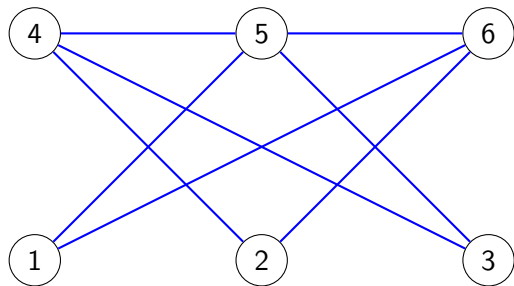


What We Know: B



C_4 : 2 – 4 – 5 – 6 – 2.

What We Know: B



C_4 : 2 – 4 – 5 – 6 – 2.

DONE!