

1 PROJECT ONE: Random Ramsey Theory

1. General Pitch 1: We know that $R(3) = 6$. But if you color the edges of K_5 at random then the prob that you DO NOT get a mono K_3 is very small. What about other Ramsey Numbers?
2. General Pitch 2: We know that $R(3) = 6$. But if you color the edges of K_6 at random then HOW MANY mono K_3 's will you get? What about other Ramsey Numbers?
3. Write a program that does the following: Given n, k
 - 1) Randomly color the edges of K_n and count the number of mono K_k 's there are.
 - 2) DO THAT 100 times (maybe less if this is to much for your computer).
4. Recall that $R(3) = 6$. That motivates the following experiment:
 - a) If you 2-color the edges of K_5 100 times Collect the data and see which number-of- K_3 's occurs the most times.
 - b) If you 2-color the edges of K_6 100 times Collect the data and see which number-of- K_3 's occurs the most times (it will be at least 1).
 - c) If you 2-color the edges of K_7 100 times Collect the data and see which number-of- K_3 's occurs the most times (it will be at least 1).
 - $\vdots$Stop at K_{10} .
Make conjectures about the following:
 - i) If you randomly color the edges of K_n then the expected number of mono K_3 's is XXX (for large n).
 - ii) If you color the edges of K_n then you will always get at least XXX mono K_3 's. (for large n).
5. Recall that $R(4) = 17$. That motivates the following experiment:
 - a) If you 2-color the edges of K_{10} 100 times Collect the data and see which number-of- K_4 's occurs the most times.
 - b) If you 2-color the edges of K_{11} 100 times Collect the data and see which number-of- K_4 's occurs the most times (it will be at least 1).
 - c) If you 2-color the edges of K_{12} 100 times Collect the data and see which number-of- K_3 's occurs the most times (it will be at least 1).
 - $\vdots$Stop at K_{25} .
Make conjectures about the following:
 - i) Find the m such that the following two hold:
If you color the edges of K_m at random then the prob of getting a mono K_4 is ≥ 0.9 .
If you color the edges of K_{m-1} at random then the prob of getting a mono K_4 is < 0.9 .

- ii) If you randomly color the edges of K_n then the expected number of mono K_4 's is XXX (for large n).
 - iii) If you color the edges of K_n then you will always get at least XXX mono K_4 's (for large n).
6. I will teach you other Ramsey Theorems that will lead to similar questions.

2 PROJECT TWO: Random VDW numbers

Same as PROJECT ONE except with VDW numbers. I give one example:

$W(3, 2) = 9$.

For all $\text{COL}[9] \rightarrow [2]$ there exists a mono 3-AP.

We suspect that most $\text{COL}[8] \rightarrow [2]$ have a mono 3-AP (though there are some that don't).

Find the smallest n such that, for all $\text{COL}[n] \rightarrow [2]$ the prob that there is a mono 3-AP is ≥ 0.9 .

3 PROJECT THREE: Ramsey Numbers via SAT Solvers

1. Investigate publicly available SAT solvers and what format they use. Try them out.
2. Phrase questions like
There is a 2-coloring of the edges of K_{13} with no mono K_4 as a SAT question. Try the SAT solvers on them.
3. Alternatively: Write your own SAT solvers and see how they do.
4. I have some other ideas about this topic as well.

4 PROJECT FOUR: Survey and Extend Cycle Results

Let C_k be the cycle of length k .

Let $R(C_k)$ be the least number such that, for all 2-colorings of the edges of K_n , there is a mono C_k .

I think that all of the numbers $R(C_k)$ are known. This survey has the references needed:

<https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1>

1. A survey of all that is known about $R(C_k)$. It might have to be about $R(C_{k_1}, C_{k_2})$.
2. $R(C_4) = 6$ is known. The proof that every $\text{COL}: \binom{[6]}{2} \rightarrow [2]$ has a mono C_4 is on my slides:
<https://www.cs.umd.edu/~gasarch/COURSES/752/S26/slides/C4talk.pdf>
 (It is not due to me.)

The proof is a lot of case work. It starts with a mono K_3 . Is there a proof with less casework than starts with two mono K_3 's?

5 PROJECT FIVE: Cycles

$R(C_4) = 6$ is known. The proof that there is a COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono C_4 is the same COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono K_3 .

Here is the odd thing: That coloring has a mono C_5 .

Note a difference between K_k 's and C_k 's:

If a graph has a mono K_k then it has a mono K_{k-1} .

If a graph has a mono C_k then it does not need to have a mono C_{k-1} .

This leads to a new problem:

$R(C_{\leq k})$ is the least n such that for all COL: $\binom{[n]}{\rightarrow}[n]$ there exists either a mono C_k or a mono C_{k+1} or $\dots$ or mono C_n .

I think this is a new notion. Study it!

6 PROJECT FOUR: Rado's Theorem for Equations that Do Not fit Rado's Theorem

Rado's Theorem Let $a_1, \dots, a_k \in \mathbb{Z}$. Let $E(a_1, \dots, a_k)$ be the equation

$$\sum_{i=1}^k a_i x_i = 0.$$

Assume that some subset of the a_i 's sums to 0. For all c , there exists $R = \text{RADO}(a_1, \dots, a_k; c)$ such that, for all COL $[n] \rightarrow [c]$ there is a monochromatic solution to $E(a_1, \dots, a_k)$. Unless the equation is $x = y$ one can make the numbers in the solution distinct.

Assume that no subset of the a_i 's sums to 0.

Let $B = \{|\sum_{i \in I} a_i| : I \subseteq [k]\}$.

Let p be the least prime that is larger than every element of B . Then there exists COL: $\mathbb{N} \rightarrow [p]$ with no mono solution.

End of Statement of Rado's Theorem

1. Consider $x + 2y - 4z = 0$. The set B is $\{1, 2, 3, 4\}$. Hence there is a coloring COL $\mathbb{N} \rightarrow [5]$ with no mono solution.

What about 2-colorings? 3-colorings? 4-colorings?

This question can be asked for any linear equation for which $0 \notin B$.

One can ask both about distinct solutions or allow non-distinct solutions.

2. There is some literature on allowing a constant, for example $x + y - z = 10$. Survey and expand it.
3. There is some literature on non-linear equations. This is very hard. Some empirical studies may work here. See next point.
4. It is known that

a) For all 2-colorings of $[7825]$ there is an x, y, z the same color with $x^2 + y^2 = z^2$.

b) There is a 2-coloring of $[7824]$ with no x, y, z the same color with $x^2 + y^2 = z^2$.

This was done with a massively big SAT solver.

QUESTION: Try to find a coloring of $[n]$ with no x, y, z etc.

You won't get anywhere near 7824, but see how far you can get.

7 Using Hard Math

1. Roth's Theorem and some other theorems use Fourier analysis. The intuition is lost on me since I don't know Fourier analysis. A paper where you do an intro to Fourier analysis the normal way (e.g., the heat equation) and then Roth's theorem so one has an idea of how Fourier analysis is being used.
2. Same with Ergodic Theory, ultrafilter proofs, topological proofs.
3. Poly VDW theorem allowing non-zero constant terms.