Notes on May 6 Meeting

Algebraic Numbers and π

1 Algebraic and Transcendental Numbers

Def 1.1

1. N is the set of natural numbers: $\{1, 2, 3, \ldots, \}$.
2. Z is the set of integers, which is the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
3. Q is the set of rationals, which is $\{\frac{a}{b} \colon a, b \in \mathsf{Z} \wedge b \neq 0\}$.
4. R is the set of reals.
5. C is the set of complex numbers which is $\{a + bi \colon a, b \in \mathsf{R}\}$.

Def 1.2

1. $\mathsf{Z}[x]$ is the set of polynomials with coefficients in Z.
2. Let $d \in \mathsf{N}$. $\mathsf{Z}[x]_d$ is the set of polynomials of degree d with coefficients in Z.

Def 1.3

1. α is *algebraic of degree 1* if it the root of a polynomial in $\mathsf{Z}[x]_1$.
2. **Exercise:** Show that a number α is rational iff α is algebraic of degree 1.
3. α is *algebraic of degree 2* if it the root of a polynomial in $\mathsf{Z}[x]_2$.
4. **Exercise:** Note that every algebraic number of degree 2 is of the form $\{a + b\sqrt{c}d$. where $a, b, c, d \in \mathsf{Z}$. Find a number of that form which is not algebraic of degree 2. Try to characterize which numbers of the form $\{a + b\sqrt{c}d$ are algebraic of degree 2.

5. An *algebraic number of degree* d is a root of an element of $\mathsf{Z}[x]_d$.

6. A number α is *algebraic* if there exists d such that α is algebraic of degree d.

7. A number α is *transcendental* if it is not algebraic.

The following are known.

1. $\sqrt{2}$ is algebraic of degree 2 since its a root of $x^2 - 2 = 0$. $\sqrt{2}$ is not algebraic of degree 1. Thats a fancy way of saying that $\sqrt{2}$ is irrational. I will prove this to you at some later point.

2. For all $d \in \mathsf{N}$, $d \geq 2$, $2^{1/d}$ is algebraic of degree d but not $d - 1$. I might prove this for you at some later point.

3. It is known that π is transcendental. This proof is difficult.

4. **Project Idea:** Prove that π is not rational using elementary means. Prove that π is not algebraic of degree 2 (or bigger) using elementary means.

5. There are many more transcendental numbers than algebraic numbers. I will make this rigorous and prove it at some later point.

2 π

In the meeting I used an inscribed and circumscribed square (a 4-gon) to show that

$$2\sqrt{2} \leq \pi \leq 4.$$

ASSIGNMENT: Use a 6-gon to get a upper and lower bound on π. Then get an 8-gon. Do these BY HAND. Once you see how to do it in general, us an n-gone.