

π Projects

Exposition by William Gasarch

May 19, 2026

Goal: Approximating π

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Motivation How to prove that $\pi \neq \frac{22}{7}$ using elementary methods.

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Second Motivation Approximating π .

Third Motivation Learn auxiliary material along the way (e.g., countability).

Inscribing/Circumscribing π Project

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Approx π by Inscribing and Circumscribing n -gons

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This approximation raises some more questions which are on the next slide.

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What to Use For $P_I(n)$, $P_C(n)$, and the average, determine the following: Given n , how many places of accuracy to π do they give?

Approximating π Using Sum of Inverse Squares

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The $\zeta(a)$'s appear in number theory and quantum mech.

Approximating π Using Other Sums

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Use FML to approximate π : How many digits can you get as you use the FML with bigger n .

π Is Not the Root Of ...

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⋮