

The $x^n + y^n$ Theorem

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Example that uses Notation!

$(\forall n \in \mathbb{N})(\exists x, y \in \mathbb{R})$

$[(x + y \in \mathbb{Z}) \wedge (x^n + y^n \in \mathbb{Z}) \wedge (x^{n+1} + y^{n+1} \notin \mathbb{Z})]$.

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For more terminology of this type see my blog post:
<https://blog.computationalcomplexity.org/2023/08/theorems-and-lemmas-and-proofs-oh-my.html>

The $(x + y, x^2 + y^2)$ Problem

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Discuss. Answer on next slide.

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(We leave that one to you)

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$$(\forall x, y \in \mathbb{R}) \\ [x + y \in \mathbb{Z} \wedge \cdots \wedge x^k + y^k \in \mathbb{Z} \implies (\forall n \in \mathbb{N})[x^n + y^n \in \mathbb{Z}]]$$

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Yes. $k = 4$ works. We prove this which is our main theorem.

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$$2a^2 = 2\left(\frac{b}{2}\right)^2 = 2\left(\frac{b^2}{4}\right) = \frac{b^2}{2} = \frac{4c^2 + 4c + 1}{4} \notin \mathbb{Z}.$$

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Since $2(xy) \in \mathbb{Z}$ and $2(x^2y^2) \in \mathbb{Z}$,
by FIRST LEMMA, $xy \in \mathbb{Z}$.

A Small Aside For A Review of Induction

Review of Induction Part I

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... We can keep doing this until we get up to $P(n)$.

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We have $P(3), P(4)$ and $P(3) \wedge P(4) \rightarrow P(5)$. So we have $P(5)$.

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$(\forall n \geq 4)[(P(n-1) \wedge P(n)) \rightarrow P(n+1)]$.

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We have $P(3), P(4)$ and $P(3) \wedge P(4) \rightarrow P(5)$. So we have $P(5)$.

We have $P(4), P(5)$ and $P(4) \wedge P(5) \rightarrow P(6)$. So we have $P(6)$.

... We can keep doing this until we get up to $P(n)$.

Back to Our Theorem

$x + y, \dots, x^4 + y^4$ Theorem

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Thm Let $x, y \in \mathbb{R}$.

$x + y, \dots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:
 $x + y, x^2 + y^2, x^3 + y^3, x^4 + y^4$.

$x + y, \dots, x^4 + y^4$ Theorem

Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:

$x + y, x^2 + y^2, x^3 + y^3, x^4 + y^4$.

Then $(\forall n \in \mathbb{N})[x^n + y^n \in \mathbb{Z}]$.

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Base $n = 1, 2, 3, 4$ are given.

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IS $x^{n+2} + y^{n+2} = (x + y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n)$.

$x + y, \dots, x^4 + y^4$ Theorem

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Thm Let $x, y \in \mathbb{R}$. Assume the following are in $\mathbb{Z}$:

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By the IH $x^{n+1} + y^{n+1} \in \mathbb{Z}$ and $x^n + y^n \in \mathbb{Z}$.

$x + y, \dots, x^4 + y^4$ Theorem

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By SECOND LEMMA $xy \in \mathbb{Z}$.

$x + y, \dots, x^4 + y^4$ Theorem

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Hence

$x^{n+2} + y^{n+2} = (x + y)(x^{n+1} + y^{n+1}) - xy(x^n + y^n) \in \mathbb{Z}$.