Euclidean Ramsey Theory Under the Manhattan Metric

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September 10th, 2025

1 Introduction

Abstract The Manhattan Metric is a concept of distance that can be applied to spaces such as $\mathbb{R}^n$. Unlike the concept of Euclidean distance given by $\sqrt{x_1^2 + ... + x_n^2}$, Manhattan distance is defined by $|x_1| + ... + |x_n|$. The goal of this paper is to study Euclidean Ramsey Theory where we use the Manhattan Metric instead of the Euclidean metric.

Keywords Euclidean Ramsey Theory; Manhattan Metric; Colorings; Chromatic Number

Notation 1.1 A c-coloring of the plane $COL : \mathbb{R}^n \to [c]$ is a function that associates each point in $\mathbb{R}^n$ one of c different colors

Notation 1.2 $d(a, b)$ - the Manhattan distance between points a and b

Notation 1.3 $\mathbb{R}^n \to (l_m, l_n)$ means that for any 2-coloring of $\mathbb{R}^n$ there exists either a red l_m or a blue l_n. (Note: for all 2-colorings we will refer to the colors as red and blue)

Def 1.4 The *chromatic Number* of the plane or χ The smallest number of colors, such that, there exists a coloring $\mathbb{R}^2 \to [\chi]$ such that for all points a, b such that if $d(a, b) = 1$ then $COL(a) \neq COL(b)$

Def 1.5 A monochromatic k cycle or C_k is a set of k points $a_1, ..., a_k$ such that set of edges between vertices are all the same color, and the first and last vertex are the same. $R(C_m, C_n)$ means you either have a monochromatic red C_m or a blue C_n

Def 1.6 A *Monochromatic Square* is 4 vertices with the same color, whose edges form right angles, and have the same distance (Note: Only the vertices themselves have to be the same color, not the segments between them)

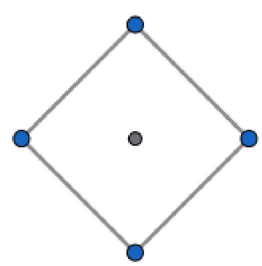
2 Chromatic Number of $\mathbb{R}^2$

Lemma 2.1 $\chi \geq 4$

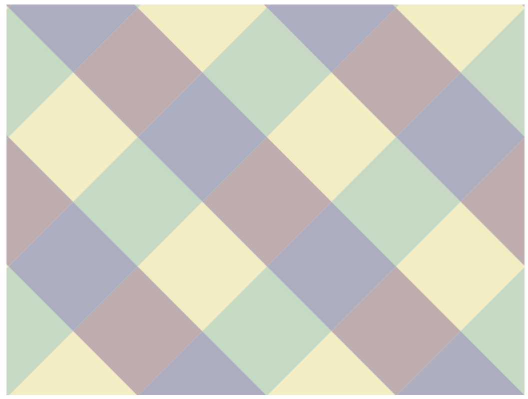
Proof:
Consider the following points $(0,0), (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (0,1)$. By the Pigeonhole Principle, if we try and 3 color 4 points then we are guaranteed that at least 2 points will share a color. Since the manhattan distance between any 2 distinct points is 1 this implies that $\chi \geq 4$.

Theorem 2.2 $\chi = 4$

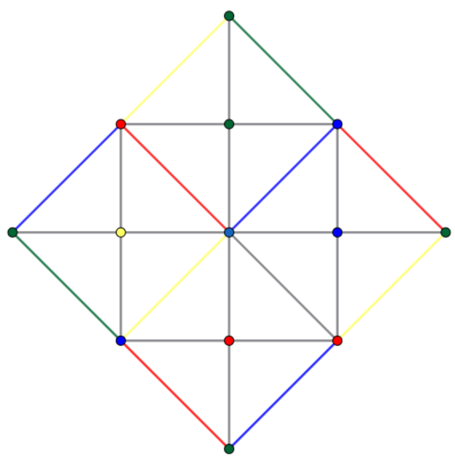
Proof:
Consider a circle under the manhattan metric.



Circles under the Manhattan metric have flat edges, and the plane can be divided into circles unlike in the Euclidean plane.



Consider the tesselation shown above. If each circle has diameter 1, then the tesselation fails only at the edges and vertices between colors. However this can be patched by alternating the colors between each edge and vertex in the following manner.

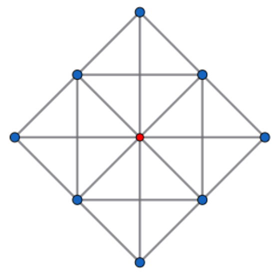


If you treat the center of each square as the entire interior being that color, then the following pattern of alternating the colors of each edge and vertex allows us to find a valid 4 coloring of the plane where no 2 points 1 inch apart are the same color. Therefore $\chi = 4$.

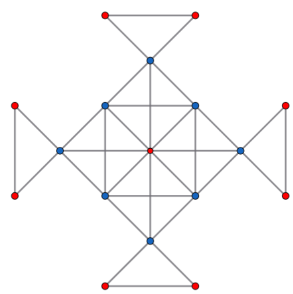
3 $\mathbb{R}^2 \rightarrow (l_m, l_n)$

Theorem 3.1 $\mathbb{R}^2 \rightarrow (l_2, l_4)$

Consider the following configuration of points:



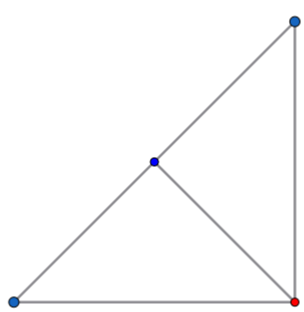
The case where all 8 points surrounding the red point are blue is the only interesting one, because every other case results in an immediate red l_2. In order to prevent a blue l_4 every collinear point a unit distance away from each blue l_3 must be red.



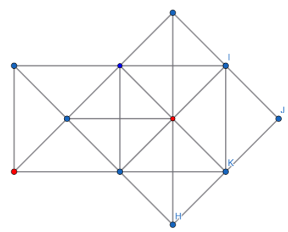
However trying to prevent a blue l_4 causes us to create 4 red l_2. Therefore, for any 2 coloring of the plane under the Manhattan Metric, we are guaranteed (l_2, l_4).

Theorem 3.2 $\mathbb{R}^2 \not\rightarrow (l_2, l_5)$

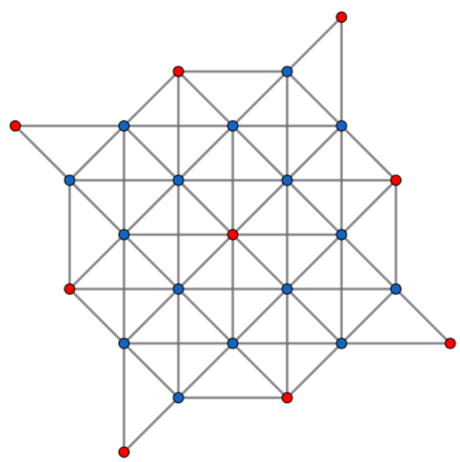
Proof:
Consider the following configuration of points:



Each line segment represents that the two points are a distance of 1 unit apart. The 3 blue points are colored the way they are, because in any other case, there would already be a red l_2. We know that on at least one side of the blue l_3 we need a red point (whichever side we choose doesn't matter, because of symmetry), and following the same logic of preventing a red l_2 this forces 2 more points to be blue.



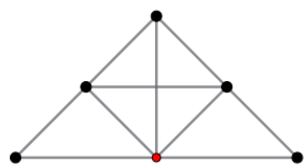
Continuing this train of logic leads us into the following symmetric pattern:



You can notice that this pattern repeats itself by looking at the red points, they all lie on the corners of a rotated square, so it can be used to completely tile the plane. This pattern has no red l_2 or blue l_5 making it a valid counterexample.

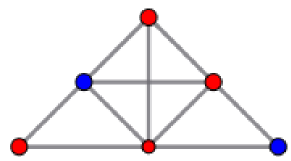
Theorem 3.3 $\mathbb{R}^2 \not\to (l_3, l_3)$

Proof: Consider this arrangement of points:

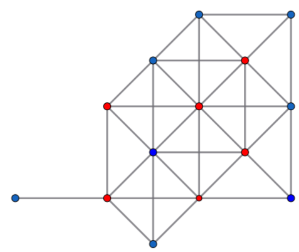


Unlike in (l_2, l_n) the points lying on the circle are not forced to be all blue, in fact they can't be, since that would result in a blue l_3. In order to prevent an l_3 of either color, each set of 3 collinear points must have at least 1 red and 1 blue. We can consider all possible starting colors and then branch off into cases. It turns out there are 4 cases (up to symmetry) that don't immediately have an l_3

Case 1:

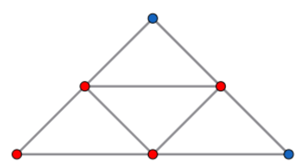


For every single red l_2 we have to make sure the points on either side have to be blue, and vice versa.

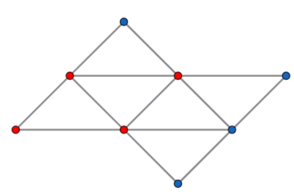


We see that following the steps does result in a blue l_3.

Case 2:

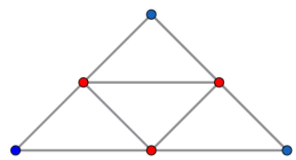


Following the same train of logic as before leads us to this configuration.

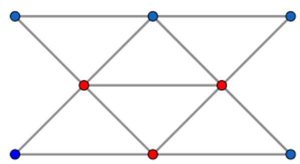


Which does have a forced blue l_3.

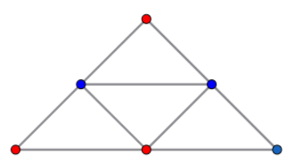
Case 3:



Leading to the following forced blue l_3.

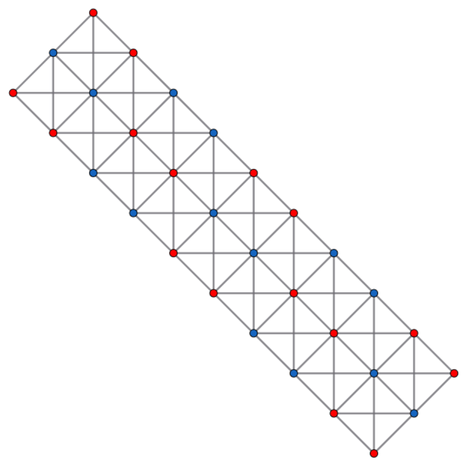


Case 4:



As noted by us going through every other possible case and finding a forced l_3 this configuration is the only one (up to rotational and translational symmetry) where an l_3 of any kind isn't forced.

The tiling shown below when extended to the entire plane avoids all possible l_3.



Therefore, there is a coloring of the plane such that there doesn't exist either a red or a blue l_3, but this coloring is unique up to rotational and translational symmetry.

(Side note: Cases that appear to be missed, such as the one with only a red point at the vertex of the triangle, actually force other cases, such as Case 3 to occur, and the same logic as the previously described 4 cases also holds if the colors are swapped.)

Open Problem 3.4 Find a function f such that $\mathbb{R}^n \to (l_3, l_{f(n)})$ and $\mathbb{R}^n \not\to (l_3, l_{f(n)+1})$.

4 Proof of Monochromatic Square in $\mathbb{R}^5$

Theorem 4.1 *For all* $COL : \mathbb{R}^6 \to [2]$ *there exists a monochromatic square*

Proof:
$p_{1,2} = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)$.
$p_{1,3} = (\frac{1}{2}, 0, \frac{1}{2}, 0, 0, 0)$.
$\vdots$
$p_{5,6} = (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$.

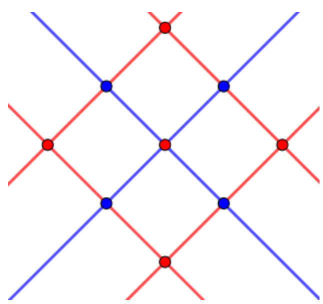
Define a new coloring $COL'(i,j) = COL(p_{i,j})$
We use the property that $R(C_4, C_4) = 6$, which guarantees any $COL : K_6 \to [2]$ to have a 4-cycle.
Since the dot product of the vectors between points is 0, we know that they all form right angles. In addition, since the distance between adjacent points is 1, this implies that any C_4 would have to be a monochromatic square. Therefore any $COL : \mathbb{R}^6 \to [2]$ contains a monochromatic square.

Theorem 4.2 *For all $COL : \mathbb{R}^5 \to [2]$ there exists a monochromatic square*

Proof:
We use the previously established points $p_{i,j}$. All of the points satisfy the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$ which is a hyperplane in $\mathbb{R}^5$. Using this, there exists a rotation and translation mapping the points $p_{i,j}$ to points $k_{i,j} \in \mathbb{R}^5$.

Theorem 4.3 *There exists a $COL : \mathbb{R}^2 \to [2]$ such that there is no monochromatic square*

Proof: Consider the counter-example:



The tiling when extended to the entire plane ensures that there are no monochromatic unit squares.

Open Problem 4.4 For all $COL : \mathbb{R}^4 \to [2]$ does there exist a monochromatic square?

Open Problem 4.5 For all $COL : \mathbb{R}^3 \to [2]$ does there exist a monochromatic square?

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