Duels with Limited Bullets

Beliz Kuter

August 2025

1 Overview

We study how bullet counts, shooting accuracies, and turn order interact to determine the outcome of a two-player duel. The duel proceeds in turns, with Alice shooting first. On their turn, if a player has bullets remaining, they will shoot; if they have no bullets, they pass. If a shot hits, the opponent is eliminated and the shooter wins. If both players run out of bullets without eliminating the other, the duel ends in a draw.

If Alice always goes first and is more accurate, but Bob has more bullets, who has the advantage? To study this question, we look for parameters where Alice and Bob are fairly evenly matched in their chances of winning. These balanced cases reveal the subtle effects of turn order and probability most clearly.

Our goal is to find the probabilities of each possible outcome:

- $P_{\text{Alice wins}}$: Alice eliminates Bob first,
- $P_{\text{Bob wins}}$: Bob eliminates Alice first,
- P_{Draw}: Both survive after all bullets are used.

The duel is defined by:

- a: number of bullets Alice starts with,
- b: number of bullets Bob starts with,
- p_A: probability Alice hits her target,
- p_B: probability Bob hits his target.

2 Our Duel Solution

Let $P(a, b)$ represent the probability that Bob survives given Alice has a bullets, Bob has b bullets, and Alice is to shoot. Let $Q(b, a)$ represent the probability that Alice survives given Bob has b bullets, Alice has a bullets, and Bob is to shoot.

We define the recurrence as follows:

Base Cases

- $P(0,0) = 1$: If both players are out of bullets, it's a draw.
- $P(0,b) = p_B \cdot 1 + (1 - p_B) \cdot P(0, b-1)$: Bob keeps shooting until he hits or runs out.
- $P(a,0) = 1 - p_A$: Alice may shoot and either hit (Bob loses) or miss (draw).

General Case

When both players still have bullets left, the duel becomes recursive because the outcome of each shot depends on what happens in the following turns.

Consider the situation $P(a,b)$, where Alice has a bullets, Bob has b bullets, and it is Alice's turn to shoot:

$$P(a,b) = (1 - p_A) \cdot (1 - Q(b, a-1))$$

This formula says:

- With probability p_A, Alice hits her shot immediately and Bob loses (so Bob's survival probability is 0 in that branch).
- With probability $(1 - p_A)$, Alice misses. Now it becomes Bob's turn, with b bullets still remaining for him and $a-1$ bullets remaining for Alice. The probability that Bob survives in this situation is given by $1 - Q(b, a-1)$, since $Q(b, a-1)$ measures Alice's survival chances when Bob is about to shoot.

Bob survives from position (a,b) only if Alice misses and then Bob manages to either win or draw in the sub-game.

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Now consider Bob's perspective with function $Q(b,a)$:

$$Q(b,a) = (1 - p_B) \cdot P(a, b-1)$$

In other words:

- With probability p_B, Bob hits immediately, and Alice is eliminated (so Alice's survival probability is 0 in that branch).
- With probability $(1 - p_B)$, Bob misses. At that point, Alice still has a bullets, Bob now has $b - 1$, and it becomes Alice's turn again. The probability of Bob surviving from that position is then $P(a, b-1)$.

Thus, $Q(b,a)$ captures Alice's survival when Bob is about to shoot and $P(a,b)$ captures Bob's survival when Alice is about to shoot. These two recurrences depend on each other, which reflects the alternating-turn nature of the duel.

3 Simulation Results

To verify our recurrence, we implemented a simulation and computed the following probabilities for different combinations of bullets and accuracies. In each scenario, Alice shoots first.

We present scenarios where Alice's and Bob's winning probabilities are very close, highlighting how small differences in bullet count or accuracy can tip the balance.

Table 1: Outcome probabilities for selected bullet counts and accuracies.

Alice Bullets	Bob Bullets	Alice Accuracy (p_A)	Bob Accuracy (p_B)	$P_{\text{Alice wins}}$	$P_{\text{Bob wins}}$	P_{Draw}
1	2	0.5	0.9	0.500	0.495	0.005
1	3	0.5	0.9	0.500	0.495	0.005
1	10	0.5	0.9	0.500	0.500	0.000
2	4	0.4	0.6	0.496	0.495	0.009
2	10	0.4	0.6	0.496	0.504	0.000
6	2	0.4	0.7	0.500	0.495	0.004
4	6	0.3	0.4	0.501	0.487	0.011
7	3	0.4	0.7	0.490	0.509	0.001
4	1	0.4	0.8	0.494	0.480	0.025
5	2	0.3	0.5	0.485	0.472	0.042
10	3	0.4	0.7	0.491	0.509	0.000

These results align with theoretical predictions and illustrate how bullet count, turn order, and shooting accuracy affect outcome probabilities.

4 Conclusion

This project has revealed the underlying mathematical structure behind a seemingly simple turn-based duel. Using recurrence relations and simulations, we were able to model and analyze how different parameters, such as bullet count, accuracy, and turn order, impact the outcome of a duel between two players.

One of the most important takeaways is the significant advantage of shooting first. Even with the same bullets and accuracy, the first shooter has a better chance to win. For example, in Table 1, Row 1 ($a = 1, b = 2, p_A = 0.5, p_B = 0.9$), Alice wins with probability 0.500 while Bob wins with 0.495, despite Bob having more bullets. This illustrates how Alice's first move provides a small but consistent edge. The same effect appeared in Row 4 ($a = 6, b = 2, p_A = 0.4, p_B = 0.7$), where Alice and Bob have similar winning chances, yet Alice still held a slight advantage from going first.

At lower accuracies, draws become more common since both players are more likely to miss. This makes the duel depend more on endurance (number of bullets) than on first-move advantage. While this does not always make Alice and Bob equally likely to win, it highlights how inaccuracy shifts the game dynamics toward survival rather than quick elimination.

Another interesting observation is that in some cases, giving Bob an additional bullet does not improve his survival chances (e.g., $(a = 1, b = 2, p_A = 0.5, p_B = 0.9)$ vs. $(a = 1, b = 3, p_A = 0.5, p_B = 0.9)$). This suggests that once Bob already has a certain margin of extra ammunition, further increases may have little effect. On the other hand, when Alice faces a slightly more accurate opponent, balancing the duel may require giving her a surprisingly large advantage in bullet count (e.g., $(a = 10, b = 3, p_A = 0.4, p_B = 0.7)$). A systematic study of how much "bullet compensation" is needed to offset accuracy differences is a promising direction for future work.

In concrete terms, Alice winning means she successfully eliminates Bob before he can respond, Bob winning means he manages to eliminate Alice before she eliminates him, and a draw means that both players miss all their shots and survive once all bullets are exhausted. These three outcomes capture the full range of possibilities in the duel and highlight how probability, turn order, and endurance interact to shape the game.

5 Future Work

A natural next step is a three-person duel. Now players must decide not just when to shoot, but also who to target. Choosing the strongest, or weakest, or forming temporary alliances can change the chances of survival, making the game more complex and interesting than the two-player duel.