Interval and Signs Projects

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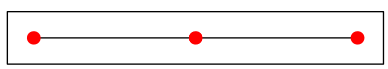
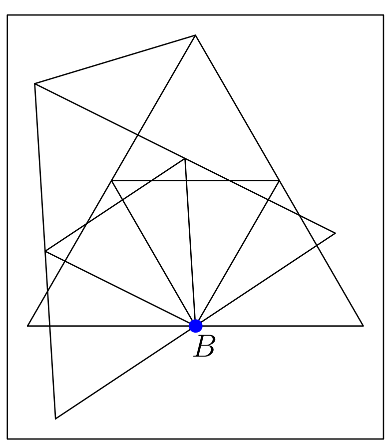
1 Abstract

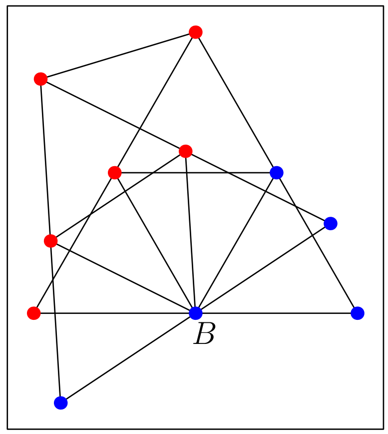
In Euclidean Ramsey Theory, a *line* is a sequence of collinear points with distance 1. Such a line is notated as an l_n where n is the number of points. For k-colorings of $\mathbb{R}^n$, a red l_n is said to be an l_n whose points are all colored red. $\mathbb{E}^n \to (l_r, l_b)$ is written to mean that in every coloring of $\mathbb{R}^n$, $\exists$ a red l_r or a blue l_b. Below is an example of a red l_3.



It is known for a few small m that $\mathbb{E}^n \to (l_3, l_m)$. The simplest example is when $m = 2$. It is trivial to prove that $\mathbb{E}^n \to (l_3, l_2)$. For example, if there are no blue points in a coloring then every l_3 is red, and if there is a blue point, the figure below shows a set of 11 points, including the point known to be blue (labeled B) for which it is impossible to color them all blue or red without getting a red l_3 or a blue l_2.



An example coloring that fails is shown below:



It has been proven by Conlon and Wu [1] that $\mathbb{E}^n \not\to (l_3, l_m)$ for some number m. Their paper used two smaller lemmas to determine that $m \leq 10^{50}$. The first lemma discussed bounds on the number of sign patterns of some quantity of polynomials with given maximum degree that accept a given number of inputs. The second lemma discussed bounds on the number of specific intervals that any quadratic falls into.

2 Introduction

Keywords Euclidean Ramsey Theory; Ramsey Theory; Sign Patterns; Intervals; Colorings

Notation 2.1 $\mathbb{E}^n$ is written to mean n-dimensional Euclidean space. A k-*coloring* of $\mathbb{R}^n$ is some $f : \mathbb{R}^n \to [k]$. In this paper we focus on 2-colorings of $\mathbb{R}^2$.

Def 2.2 Let $\mathrm{Reg}_{\max}(d, n, p)$ be the *maximum number of regions n polynomials with maximum degree p with d inputs.*

Def 2.3 Let $\mathrm{Reg}_{\mathrm{maxlin}}(d, n, p)$ be the *maximum number of regions n linear polynomials with maximum degree p with d inputs.*

Def 2.4 Let l_n be a set of points $a_1, a_2, ..., a_n$ where $a_{i+1} - a_i = u$ for some $|u| = 1$.

Def 2.5 A polynomial $P(a_1, a_2, ..., a_d) = \sum_{i_1,i_2,...,i_d=0}^{p} C_{i_1,i_2,...,i_d} \cdot a_1^{i_1} \cdot a_2^{i_2} \cdot ... \cdot a_d^{i_d}$ for constant c.

Def 2.6 A linear polynomial $O(a_1, a_2, ..., a_d) = \sum_{i=1}^{d} c_i a_i$ for constant c.

Notation 2.7 Let the set $\mathbb{P}_{d,p}$ consist of all P with d inputs and maximum degree p. Let the set $\mathbb{P}_{d,n,p}$ be the power set of $\mathbb{P}_{d,p}$.

It will later be useful to consider trivial cases of Reg_{max} and of $\text{Reg}_{\text{maxlin}}$. Some are listed here.

An obvious bound on $\text{Reg}_{\text{max}}(d, n, p)$ is 3^n. For n polynomials, each may return one of three signs: $+$, $-$, or 0. Therefore, there exist 3^n sign patterns.

$\text{Reg}_{\text{maxlin}}(0, n)$ represents a set of constants. Since the sign of a constant never changes, the sign pattern is always the same, so $\text{Reg}_{\text{maxlin}}(0, n) = 1$.

$\text{Reg}_{\text{maxlin}}(d, 0)$ represents exactly zero polynomials. Since there are no polynomials, the sign pattern is of length 0, for which only one pattern exists. Therefore, $\text{Reg}_{\text{maxlin}}(d, 0) = 1$.

Def 2.8 A function $F(n) = \text{mod}(n^2 + \alpha n + \beta, p)$ for real α,β and prime p. $F(n)$ then takes on values in $[0, p)$.

A function $R(n)$ is taken to be to be the number of regions $[0, 1), [1, 2), ..., [p - 1, p)$ that contain at least one value in $F(1), F(2), ..., F(n)$. For the sake of this paper, F will be defined in advance.

Def 2.9 $R(n, \alpha, \beta) = |\{i| \, |\{j | j \in [n], i \leq F(j) < i + 1\}| \neq 0\}|$ for $F(n) = \text{mod}(n^2 + \alpha n + \beta, p)$.

It will also be useful to consider the inverse of R, which we will call Q.

Notation 2.10 Let $Q(\alpha, \beta)$ be the smallest number such that $R(Q(\alpha, \beta), \alpha, \beta) = p$, if such a number exists at all.

Our proof focuses on a geometric interpretation of $\text{Reg}_\text{m}\text{ax}$, where Olenik's mainly relied on matrix manipulation.

In Section 1 we present the problem and a basic overview of Conlon and Wu's paper. In Section 2 we state definitions regarding the two lemmas given in Conlon and Wu's paper. In Section 3 we discuss and attempt to improve the result given by Olenik. In Section 4 we use a Monte Carlo simulation to give a better bound on the number of regions a polynomial falls into. In Section 5 we use our results to improve the bounds on m and present open questions.

3 Signs

Earlier, it was shown that $\text{Reg}_{\max}(d,n,p)$ is bounded by 3^n. However, a specific case of the Olenik-Petrovsky theorem that $\text{Reg}_{\max}(d,n,p) \leq \left(\frac{50pn}{d}\right)^d$ for $n > d$. [2]

Often, it is useful to imagine the sign of a function P with d inputs as a surface cutting $\mathbb{R}^d$ into at most parts: where $P > 0$, $P < 0$, and where $P = 0$. Note that some polynomials, like x^2 or x^2+1, do not divide space into three pieces.

For a set of polynomials $P_1, P_2, ..., P_n$, the $\mathbb{R}^{d-1}$ surface $P_i = 0$ splits $\mathbb{R}^d$ into some number of regions. (For the sake of this paper, we take $P = 0$ as a region as well as $P > 0$ and $P < 0$.) Due to the continuity of polynomials, every region contains only points with the same sign pattern. Therefore, the number of sign patterns $\leq \text{Reg}_\text{m}\text{ax}(d,n,p)$.

Lemma 3.1 *The intersection of any two hyperplanes in $\mathbb{R}^n$ is a hyperplane in $\mathbb{R}^{n-1}$.*

Proof: Let H be the hyperplane defined by the set of points $\overrightarrow{x}$ where $\overrightarrow{h_c} \cdot \overrightarrow{x} + h_i = 0$, and J be the set of points $\overrightarrow{x}$ where $\overrightarrow{j_c} \cdot \overrightarrow{x} + j_i = 0$. Then let $h_j = \sum_{i=2}^{n} h_{c,i}x_i + h_i$ and $j_j = \sum_{i=0}^{n-2} j_{c,i}x_i + j_i$. Then $h_j + h_{c,0}x_0 + h_{c,1}x_1 = 0$ and $j_j + j_{c,0}x_0 + j_{c,1}x_1 = 0$. If $\begin{bmatrix} h_{c,0} & h_{c,1} \\ j_{c,0} & j_{c,1} \end{bmatrix}$ is invertible, then there is one solution for x_0 and x_1. ∎

Lemma 3.2 $\text{Reg}_{\text{maxlin}}(d, n+1) \leq \text{Reg}_{\text{maxlin}}(d,n) + 2\,\text{Reg}_{\text{maxlin}}(d-1,n)\, \forall d, n > 1$.

Proof: Assume you have n hyperplanes, with d inputs, that divide space into the maximum number of regions, e.g. $\mathrm{Reg}_{\mathrm{maxlin}}(d,n)$ regions. Suppose you add another hyperplane H, defined by $P=0$. This hyperplane may intersect any number of other planes. The intersections of each hyperplane with the new hyperplane form a hyperplane with one fewer dimension. At worst, you will get $\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)$ regions formed by the intersection. Notice that these regions exist in the higher level of space, where $P=0$. Every region that existed before H that is split by H becomes either two or three new regions. All points in this region will have the same sign pattern, except for the sign of P, which might be $1,0$,or-1 for those points. We may assume the cut makes three new regions to calculate a bound for $\mathrm{Reg}_{\mathrm{maxlin}}$. Therefore, you get 2 more regions then there were before for those divided. Recall that since $\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)$ regions were formed, $\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)$ regions must have been cut. Therefore, you have $\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)\cdot 2$ new regions. There were $\mathrm{Reg}_{\mathrm{maxlin}}(d,n)$ regions before this new hyperplane was added, so after adding another hyperplane you have $\mathrm{Reg}_{\mathrm{maxlin}}(d,n)+2\,\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)$ regions. However, the new hyperplane represents another polynomial with degree 1 and d dimensions, so the number of new regions is, at worst, $\mathrm{Reg}_{\mathrm{maxlin}}(d,n+1)$. Therefore, $\mathrm{Reg}_{\mathrm{maxlin}}(d,n+1)\leq \mathrm{Reg}_{\mathrm{maxlin}}(d,n)+2\,\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)$. ∎

Lemma 3.3 $\mathrm{Reg}_{\mathrm{maxlin}}(d,n)\leq \sum_{i=0}^{d} 2^i \binom{n}{i}$.

Proof: Let $R(d,n)=\sum_{i=0}^{d} 2^i\binom{n}{i}$. R has the unique property that $R(0,n)=R(d,0)=1$, and that $R(d,n)=R(d,n-1)+2R(d-1,n-1)$. Thus for $d=0$ or $n=0$, $\mathrm{Reg}_{\mathrm{maxlin}}(d,n)=R(d,n)$ Assume it is known that $\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)\leq\sum_{i=0}^{d-1}2^i\binom{n}{i}$ and that $\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n-1)\leq\sum_{i=0}^{d-1}2^i\binom{n-1}{i}$. Then $\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)+2\,\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n-1)\leq R(d-1,n)+2R(d-1,n-1)=R(d,n)$. But by the previous lemma $\mathrm{Reg}_{\mathrm{maxlin}}(d,n)\leq \mathrm{Reg}_{\mathrm{maxlin}}(d-1,n)+2\,\mathrm{Reg}_{\mathrm{maxlin}}(d-1,n-1)$. Then $\mathrm{Reg}_{\mathrm{maxlin}}(d,n)\leq R(d,n)$. ∎

Lemma 3.4 $\mathrm{Reg}_{\mathrm{max}}(d,n,p)\leq \mathrm{Reg}_{\mathrm{maxlin}}(d^p,n)$.

Proof: Every polynomial in $\mathbb{P}_{(d,n,p)}$ is equal to $\sum_{i_1,i_2,\ldots,i_d=0}^{p} C_{i_1,i_2,\ldots,i_d}\cdot a_1^{i_1}\cdot a_2^{i_2}\cdot\ldots\cdot a_d^{i_d}$ for some $C_{i_1,i_2,\ldots,i_d}$ and inputs $a_1,a_2,\ldots,a_d$. Then view the polynomial as a linear polynomial with inputs $a_1^{i_1}\cdot a_2^{i_2}\cdot\ldots\cdot a_d^{i_d}$. Assuming all values of $a_1^{i_1} \cdot a_2^{i_2} \cdot ... \cdot a_d^{i_d}$ are independent, we can treat them as another set of inputs, $a'_1, a'_2, ..., a'_{d^p}$, and treat C as another list of constants $C'_{1,2,...,d^p}$. Therefore every polynomial can be written as $\sum_{i=0}^{d^p} C'_i \cdot a'_i$. However, $a'_1, a'_2, ..., a'_{d^p}$ may or may not take on any set of values, since they are all functions of a smaller set of variables, $a_1, a_2, ..., a_d$. So it is indeed the case that $\mathrm{Reg}_{\max}(d, n, p) \leq \mathrm{Reg}_{\mathrm{maxlin}}(d^p, n)$. ∎

These lemmas give an upper bound of $\mathrm{Reg}_{\max}(d, n, p) \leq \sum_{i=0}^{d^p} 2^i \binom{n}{i}$. In the context of Conlon and Wu's proof, it was useful to consider the case when $n = 4m^3, p = 1, d = 2$, where $\mathbb{E}^n \not\to (l_3, l_m)$. The form provided by Olenik et. al. gives $\mathrm{Reg}_{\max}(2, 4m^3, 1) \leq 10^4 m^6$, but the upper bound discussed here gives $\mathrm{Reg}_{\max}(2, 4m^3, 1) \leq \sum_{i=0}^{2^1} 2^i \, \mathrm{nCr}(4m^3, i) = 1 + 32m^6$.

Lemma 3.5 $\mathrm{Reg}_{\max}(d, n, p) = 3^n$ *if* $d \geq n$.

Proof: Consider the polynomials $p_1 = a_1$, $p_2 = a_2$, $p_3 = a_3$, $. \; . \; . \; p_n = a_n$ (and since $n > d$, it is obvious that such polynomials can be defined.) Then the sign pattern of $p_1, p_2, ..., p_n$ is the same as the sign pattern of $a_1, a_2, ..., a_n$. The values of $-1, 0, 1$ have signs $-, 0, +$, respectively, so any sign pattern of $p_{1,...,n}$ is achievable via setting each of $a_{1,...,n}$ to $-1, 0$, or 1, as desired. ∎

Lemma 3.6 $\mathrm{Reg}_{\max}(d, n, p) \geq 3^d$ *if* $d < n$.

Proof: Consider the polynomials $p_1 = a_1$, $p_2 = a_2$, $p_3 = a_3$, $. \; . \; . \; p_d = a_d$, $p_{d+1} = 0$, $p_{d+2} = 0$, $. \; . \; . \; p_n = 0$. Then the sign pattern of $p_1, p_2, ..., p_d$ is the same as the sign pattern of $a_1, a_2, ..., a_d$, and the remaining polynomials have sign pattern 0. The values of $-1, 0, 1$ have signs $-, 0, +$, respectively, so any sign pattern of $p_{1,...,d}$ is achievable via setting each of $a_{1,...,d}$ to $-1, 0$, or 1, as desired. Therefore there are at least 3^d sign patterns given. ∎

Lemma 3.7 $\mathrm{Reg}_{\max}(d, n, p) \geq 2^n$ *for large* d.

Proof: Consider the sequence of functions $P_i(a_1, a_2, ..., a_d) = \left(\sin\left(2^i \pi a_1 - \frac{\pi}{4}\right)\right)^2 - 0.5$ for $i = 0, 1, ..., n-1$. The sign of $P_i(a_1, a_2, ..., a_d) =$

$$\begin{cases} -1 & 0 < \mathrm{mod}(a_1, 2^{-i}) < 0.5 \cdot 2^i \\ 1 & \mathrm{mod}(a_1, 2^{-i}) > 0.5 \cdot 2^{-i} \\ 0 & \mathrm{mod}(a_1, 2^{-i}) = 0.5 \cdot 2^{-i} \\ 0 & \mathrm{mod}(a_1, 2^{-i}) = 0. \end{cases}$$

Then consider the list of points $L =$ for $i = 0, 1, 2, ..., 2^n - 1$.

Lemma 3.8 *No two points in L will have the same sign pattern over* $P_0, P_1, ..., P_{(n-1)}$.

Case 1. At least one point in L has a 0 in their sign pattern. This means that $\bmod(x, 2^{-i}) = 0.5 \cdot 2^{-i}$ or 0. for some x and i. This means that $0.25 \cdot 2^{-n} + m \cdot 0.5 \cdot 2^{-n} = 0.5 \cdot 2^{-i} + k \cdot 2^{-i}$ for some $m, k \in \mathbb{N}$, and i. Multiplying both sides by $\cdot 2^{i+1}$ gives $(m + 0.5) \cdot 2^{-n+i} = 2k + 1$. Since $n > i$, the left side is not an integer, but the right side is.

Case 2. No points in L have a 0 in their sign pattern. Assume points m_1, m_2 have the same sign pattern, but that $m_1 \neq m_2$. WLOG assume $0 < m_1 < m_2 < 1$. Since $\text{SIGN}(P_0(m_1)) = \text{SIGN}(P_0(m_1))$, either $0 < m_1 < m_2 < 0.5$ or $0.5 < m_1 < m_2 < 1$. In the first case, map m_1, m_2, and $P_{0,1,2,...,n-1}$ with the transform $x \to 2x - \frac{1}{2^{n+1}}$. Notice under this transform, $P_{i+1} \to P_i$ and $\left[\frac{1}{2^{n+1}} + i \cdot \frac{1}{2^n}\right] \to$ some subset of $\left[\frac{1}{2^n} + i \cdot \frac{1}{2^{n-1}}\right]$. In the second case, the transform $2x - 1 - \frac{1}{2^{n+1}}$ has the same property. So if $\text{Reg}_{\max}(d, n-1, p) \geq 2^{n-1}$, $\text{Reg}_{\max}(d, n, p) \geq 2^n$. When $n = 1$, $P_0(0.25, 0, ...) > 0$ and $P_0(0.75, 0, ...) < 0$, giving $2^1 = 2$ sign patterns.

Then we can make a polynomial s(x) = sin(x). The taylor series of sin(x) converges everywhere, meaning for a large enough maximum degree n, sin(x) will deviate by no more than some epsilon. Since the points of L give sign patterns without 0, for some epsilon small enough, the sign patterns of all points in L will stay the same.

The nth term of the taylor series for sin(x) is $(-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Since it alternates, we want to find the smallest d for which $\frac{x^{2d+1}}{(2d+1)!} \leq \varepsilon$ with x. Such a value must exist, therefore 2^n is an error bound on $\text{Reg}_{\max}$. ∎

4 Intervals

The interval problem is more difficult. It is known $F(1), F(2), ..., F(p^3)$ mod p must fall into at least $\frac{p}{6}$ regions for $F = x^2 + \alpha x + \beta$ with $\alpha, \beta \in \mathbb{R}$. However, it often falls into a greater number of regions. One can make a graph of the smallest number of values of F required to have each region contain some value of F. Below is such a graph, with $\alpha \in [0, 1]$ along the x-axis and $\beta \in [0, 1]$ along the y-axis. In other words, the graph is of $Q(\alpha, \beta)$ for $p = 7$. Interestingly, there seem to be polygonal regions with widely varying values instead of something smooth and [not straight]. A similar graph can be made for larger values of p.

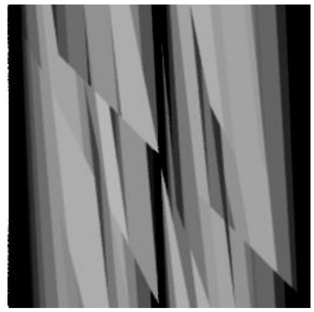


Figure 1: Graph of interval count for p=7. Darker values indicate a larger Q, while bright white indicates Q=0.

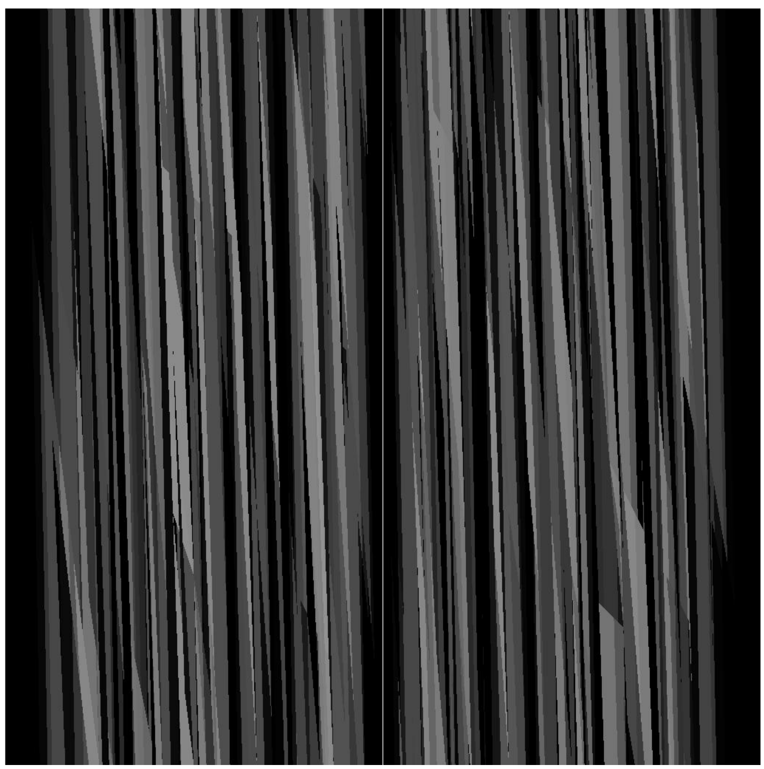


Figure 2: Graph of interval count for p=17. Darker values indicate a larger Q.

A Monte Carlo simulation was run to evaluate $Q(\alpha, \beta)$ over $0 \leq \alpha < 1, 0 \leq \beta < 1$. Specifically, the region $0 \leq \alpha < 1, 0 \leq \beta < 1$ was divided into 128^2 equal square regions. Then, for each smaller region, 64 values of α and β were picked in the region, and Q was evaluated. The distribution of Q is shown below.

p	Average	Min	Lower Quartile	Mode	Upper Quartile	Max
7	22.77	9	14.92	14.01	22.29	124.54
11	51.52	18.96	35.13	31.33	50.89	310.13
13	72.72	29.91	55.45	45	71.53	406.2
17	78.34	42.77	50.91	54.69	61.37	399.62
19	112.05	46.57	58.96	72.4	125.96	561.72
23	134.19	63.23	75.96	88.43	153.32	722.8
29	147.86	80.98	93.78	111.02	167.65	733.69
31	167.13	88.76	108.88	115.78	175.66	839.29
37	212.45	114.3	141.97	159.77	228.24	1053.36
41	245	127.08	159.8	177.3	240.42	1194.09
43	272.51	142.88	170.54	192.66	283.93	1334.18
47	263.07	167.74	188.4	201.49	285.29	1253.69

From this data we see that the maximum value of Q grows roughly linearly with p. In fact, for this set of data, $Q \approx 27p$. So, for $m = 27p$, roughly p intervals are hit.

The proof itself uses the polynomial $a + d(i-1) + i^2 - 3i + 2$ for some a, d, and prime p.

5 Conclusions and Open Problems

Conlon and Wu's result required $12p^{-\frac{1}{4}} < \frac{1}{2}$ and $\text{Reg}_{\max}(2, Q(p), 1)\left(1 - p^{-\frac{3}{4}}\right)^p < \frac{1}{2}$, where $m = Q(p)$. This gives $10^9 < p$, for which there exists at least one prime $\leq 2 \cdot 10^9$. Finally, since $m = 27p$, $\exists m \leq 54 \cdot 10^9 | \mathbb{E}^2 \to (l_3, l_m)$, and, equivalently, $\mathbb{E}^2 \to (l_3, l_{10^{10}})$. Without using the results shown here for the interval problem, we get the inequality $(32m^6 + 1)\left(1 - p^{-\frac{3}{4}}\right)^{\frac{p}{6}} \leq \frac{1}{2}$, for $m = p^3$.

6 Acknowledgements

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References

[1] A. Einstein. More on lines in Euclidean Ramsey theory. (German) [On the electrodynamics of moving bodies]. *Annalen der Physik*, 322(10):891–921, 1905.

[2] A. Einstein. Zur Elektrodynamik bewegter Körper. (German) [On the electrodynamics of moving bodies]. *Annalen der Physik*, 322(10):891–921, 1905.