

The Circle-Crossing Game

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1 The Circle-Crossing Game

In this paper we study the Circle-Crossing Game, defined as follows.

The Circle-Crossing Game. Let $n \geq 4$. Place n points on a circle. The players are Player One and Player Two, with Player One moving first. On each turn, a player draws a chord between two points for which a chord has not already been drawn. The first player who draws a chord that intersects an existing chord loses.

Two questions arise:

1. For which values of n does Player One win, and for which values does Player Two win?
2. For the winning player, what is the winning strategy?

We prove the following.

Theorem 1. *Let $n \geq 4$. Player One has a winning strategy. The strategy is simple: on every turn, draw a chord that does not intersect any chord already on the board.*

The proof rests on a counting fact about triangulations of convex polygons.

Lemma 1. *Every maximal set of pairwise noncrossing chords on n points arranged on a circle contains exactly $2n - 3$ chords.*

Proof. A boundary edge, meaning a chord connecting two adjacent points on the circle, cannot cross any other chord. Therefore, every maximal set of noncrossing chords must contain all n boundary edges.

In a maximal noncrossing configuration, every region must be a triangle. If some region had four or more sides, another diagonal could be drawn inside that region without crossing any existing chord, contradicting maximality. Thus, a maximal noncrossing configuration is exactly a triangulation of a convex n -gon.

A triangulation of a convex n -gon contains exactly $n - 2$ triangles and $n - 3$ interior diagonals. One way to prove this is by induction: split the polygon along a diagonal into two smaller polygons and add the triangle and diagonal counts from the two pieces. A second proof follows from Euler's formula for planar graphs, $V - E + F = 2$, applied to the triangulation.

Therefore, the total number of chords in a maximal noncrossing configuration is

$$n + (n - 3) = 2n - 3.$$

End of Proof

We now restate and prove the main theorem.

Theorem 2. *Let $n \geq 4$. Player One has a winning strategy: on every turn, draw a chord that does not intersect any chord already on the board.*

Proof. By the lemma, every maximal noncrossing configuration contains exactly $2n - 3$ chords. Since $2n - 3$ is odd for every integer n , Player One makes the last safe move whenever both players continue drawing noncrossing chords.

If Player Two draws a crossing chord earlier, Player Two loses immediately. Otherwise, after Player One draws the $(2n - 3)$ rd safe chord, the board is a maximal noncrossing configuration. No additional chord can be drawn without crossing an existing chord. Player Two must therefore draw a crossing chord and loses. Hence Player One has a winning strategy for every $n \geq 4$. *End of Proof*

2 Computational Verification

Once we had the proof, we still wanted to check it against actual simulated games rather than just trusting the combinatorics, partly because it is such a clean and almost too convenient result. I built a general game-playing framework in Python that could simulate the Circle-Crossing Game for any n , with both players choosing legal moves uniformly at random from whatever chords were still available and noncrossing.

We ran 10,000 independent simulated games for each of $n = 6, 7, 8$, and 9 , for 40,000 games total. Player One won every single simulated game across all four board sizes, with zero exceptions, which lined up exactly with what the theorem predicted. We stopped at $n = 9$ because that range was enough for the computational check, while the runtime increased enough that simulations for $n = 10$ were too slow to be useful for our purposes.

Since the proof does not rely on any particular value of n or any upper bound, the result extends to every board size beyond what we actually simulated. Running the simulations was not necessary to trust the proof once we had it fully worked out, but it was a useful sanity check while we were still building the argument, since a single simulated counterexample at any point would have told us we had made a mistake somewhere in the triangulation logic.

3 Reflection

Going into the summer, I expected the harder-looking games in the project to also be the harder ones to understand, and the Circle-Crossing Game broke that assumption for me. It looks like the simplest game out of everything the group studied, points and chords and nothing else. In one sense, the final result is simple: Player One wins as long as the player avoids making a bad move. But proving that result was more complicated than the game first made it seem. It required learning new mathematics, working carefully through the triangulation count, and making sure the parity argument was completely airtight. I think that was the biggest thing I took away from the summer: how complicated a problem looks and how complicated it actually is can be two completely different

things, and you do not find out which one you are dealing with until you actually sit down and try to prove something about it.

The part that took me the longest to really get was the triangulation counting argument. I could see from small examples that the chord count kept coming out to $2n - 3$, but it took a while working through the induction with the rest of the team before it stopped feeling like a coincidence and started feeling like something we actually understood. Redoing the count a second time with Euler's formula and getting the same number was probably the most satisfying moment of the summer for me, since it meant we were not just fooling ourselves with one shaky argument.

Working alongside the rest of the team also changed how I think about splitting up a proof. None of us worked out the whole thing alone; one person would push on the boundary-edge argument, someone else would work through the induction, and I ended up handling most of the simulation code to check our proof against actual games. Watching a proof come together from a few different angles at once was different from any math I had done before, where usually you are just working through a problem by yourself until it is done.

Professor Gasarch was the one who kept pushing us not to stop at the empirical pattern. It would have been easy to run a few thousand simulations, see Player One win every time, and just report that as the finding. He kept asking us why it had to be true, which is what actually got us to the triangulation argument in the first place. I think that is the habit I am most hoping to carry forward, treating a strong pattern in the data as a question to answer rather than an answer in itself.

4 Acknowledgments

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