0.1 The Half method

On the slides we proved \( f(11,5) \leq \frac{13}{30} \). A brief review:

1. Since \( \frac{13}{30} > \frac{1}{3} \), every muffin is cut into 2 pieces, so there are \( 2m \) pieces.
2. Since each muffin is cut into 2 pieces that are buddies, there are at most 11 pieces that are \( > \frac{1}{2} \).
3. We showed that any procedure with smallest piece \( > \frac{13}{30} \)
   would have at least 12 shares \( > \frac{1}{2} \). This gave a contradiction.

We generalize this technique, which we call The Half method.
It works just as well if we end up with more than \( m \) shares \( < \frac{1}{2} \).

There are many cases of the Half method. Therefore we give
2 more examples of what can happen when it is applied:

- \( f(45,26) \leq \frac{32}{78} \)
- \( f(29,17) \leq \frac{27}{68} \)

0.2 \( f(45, 26) \leq \frac{32}{78} \) by The Half method

**Theorem 0.1.** \( f(45, 26) \leq \frac{32}{78} \).

**Proof.** Assume, by way of contradiction, that there is a \((45, 26)\)-procedure with smallest piece \( > \frac{32}{78} \). Since \( \frac{32}{78} > \frac{1}{3} \) every muffin is cut into exactly 2 pieces. Hence there are 90 pieces. Note that there can be at most 45 pieces \( < \frac{1}{2} \). We show that there is a piece \( \leq \frac{32}{78} \).

Every student gets \( \frac{45}{26} = \frac{45 \times 3}{26 \times 3} = \frac{135}{78} \).

**Case 1:** Alice gets \( \geq 5 \) shares. Then one of them is \( < \frac{135}{78} \times \frac{1}{5} = \frac{27}{78} \).

**Case 2:** Bob gets \( \leq 2 \) shares. Then one of the shares is \( \frac{135}{78} \times \frac{1}{2} = \frac{67.5}{78} \). Its buddy is \( < 1 - \frac{67.5}{78} = \frac{10.5}{78} < \frac{32}{78} \).

In the subsequent cases we assume the negation of Cases 1 and 2. Hence everyone is either a 3-student or a 4-student. Let
Let $s_3$ (s_4) be the number of 3-students (4-students). Since there are 90 pieces and 26 students,

$$3s_3 + 4s_4 = 90$$
$$s_3 + s_4 = 26.$$ 

Hence $s_3 = 14$ and $s_4 = 12$. So there are fourteen 3-students, twelve 4-students, forty-two 3-shares, and forty-eight 4-shares. Since $48 > 45$, if all of the 4-shares are $< \frac{1}{2}$, that will be a contradiction. Indeed, this will be our contradiction.

We now look at intervals.

**Case 3:** Alice has a 4-share $\geq \frac{39}{78}$. Alice’s other three 4-shares add up to $\leq \frac{135}{78} - \frac{39}{78} = \frac{96}{78}$, hence one of them is $\leq \frac{96}{78} \times \frac{1}{3} = \frac{32}{78}$. 

**Case 4:** Bob has a 3-share $\leq \frac{43}{78}$. Bob’s other two 3-shares add up to $\geq \frac{135}{78} - \frac{43}{78} = \frac{92}{78}$, hence one of the shares is $\geq \frac{92}{78} \times \frac{1}{2} = \frac{46}{78}$. Its buddy is $\leq 1 - \frac{46}{78} = \frac{32}{78}$.

**Case 5:** The following picture captures the negation of cases 1,2,3, and 4.

$$\begin{array}{c|c|c|c|c}
(48 \text{ 4-shs}) & 0 & (42 \text{ 3-shs}) \\hline
\frac{32}{78} & \frac{39}{78} & \frac{43}{78} & \frac{46}{78}
\end{array}$$

The midpoint is $\frac{1}{2} = \frac{39}{78}$. Note that all forty-eight 4-shares are $< \frac{1}{2}$. This is a contradiction.

We show how one could derive the upper bound $f(45,26) \leq \frac{32}{78}$. Let $\alpha$ be the upper bound. We derive conditions on $\alpha$ that will make the proof of $f(45,26) \leq \alpha$ work. We assume $\alpha > \frac{1}{3}$.

We guess everyone is either a 3-student or a 4-student.)

In the proof that $f(45,26) \leq \frac{32}{78}$ we deduced that there are forty-two 3-shares and forty-eight 4-shares. This calculation did not use that the goal was $\frac{32}{78}$. Hence we can use that reasoning. We have the following picture, though we do not know $x$ or $y$.
What are $x$ and $y$?

- $x$ is the least number such that every 4-share is $< x$. Hence $3\alpha + x = \frac{135}{78}$, so $x = \frac{135}{78} - 3\alpha$.
- $y$ is the largest number such that every 3-share is $> y$. Hence $2(1 - \alpha) + y = \frac{135}{78}$, so $y = 2\alpha - \frac{7}{26}$.

Hence we have:

\[
\begin{array}{cccc}
(48 \text{ 4-shs}) & 0 & (42 \text{ 3-shs}) \\
\alpha & x & y & 1 - \alpha
\end{array}
\]

If $x \leq \frac{1}{2} \leq y$ then there will be 48 > 45 shares to the left of \(\frac{1}{2}\) which is a contradiction. We look at setting $x = \frac{1}{2}$ and $y = \frac{1}{2}$.

If $x = \frac{1}{2}$ then

\[\alpha = \frac{135}{78} - \frac{1}{2} = \frac{16}{39}.\]

If $y = \frac{1}{2}$ then

\[\alpha = \frac{1}{2} + \frac{7}{26} = \frac{5}{13}.\]

You would think we should take the lower value, $\alpha = \frac{5}{13}$. But, alas, if you try to do the proof with this value you get that $y < x$ so the proof would not work. Hence we take $x = \frac{16}{39}$. 
0.3 \( f(29, 17) \leq \frac{27}{68} \) by The Half method

In the proof of Theorem 0.1, the intervals containing the 3-shares and the intervals containing the 4-shares did not overlap. (This is the most common case for the Half method.) Is there a case where the intervals overlap and the Half method still works? Yes. We present one.

**Theorem 0.2.** \( f(29, 17) \leq \frac{27}{68} \).

**Proof.** Assume, by way of contradiction, that there is a \((29, 17)\)-procedure with smallest piece \( \frac{27}{68} \). Since \( \frac{27}{68} < \frac{1}{3} \) every muffin is cut into exactly 2 pieces. Hence there are 58 pieces. Note that there can be at most 29 pieces \( \frac{1}{2} \).

Every student gets \( \frac{29}{17} = \frac{29 \cdot 4}{17 \cdot 4} = \frac{116}{68} \).

We leave as an exercise to show that (1) if Alice has \( \geq 5 \) shares then she has a share \( \leq \frac{27}{68} \), (2) if Bob has \( \leq 2 \) shares then one of them has a buddy that is \( \leq \frac{27}{68} \), hence (3) everyone is a 3-student or a 4-student, and (4) there are ten 3-students, seven 4-students, thirty 3-shares, and twenty-eight 4-shares. Since \( 30 > 29 \), if all of the 3-shares are \( > \frac{1}{2} \), that will be a contradiction. Indeed, this will be our contradiction.

We now look at intervals.

**Case 1:** Alice has a 4-share \( \geq \frac{35}{68} \). Alice’s other three 4-shares sum to \( \leq \frac{116}{68} - \frac{35}{68} = \frac{81}{68} \), hence one of them is \( \leq \frac{81}{68} \times \frac{1}{3} = \frac{27}{68} \).

**Case 2:** Bob has a 3-share \( \leq \frac{34}{68} \). Bob’s other two 3-shares sum to \( \geq \frac{116}{68} - \frac{34}{68} = \frac{82}{68} \), hence one of the shares is \( \geq \frac{82}{68} \times \frac{1}{2} = \frac{41}{68} \). Its buddy is \( \leq 1 - \frac{41}{68} = \frac{27}{68} \).

**Case 3:** The negation of cases 1 and 2. I know what you are thinking. We’ll just draw the picture and have a good sense of what is going on. But the picture is hard to draw. Why? Let’s draw the 4-share and 3-share pictures separately.

**The 4-shares:**
( 28 4-shs ) ( 0 4-shs )
\[ \begin{array}{ccc}
\frac{27}{68} & \frac{35}{68} & \frac{41}{68}
\end{array} \]

The 3-shares:

( 0 3-shs ) ( 30 3-shs )
\[ \begin{array}{ccc}
\frac{27}{68} & \frac{34}{68} & \frac{41}{68}
\end{array} \]

They overlap. The interval \( (\frac{34}{68}, \frac{35}{68}) \) can contain both 3-shares and 4-shares. Can our proof proceed anyway? Yes.

All thirty 3-shares are bigger than \( \frac{1}{2} \). This is a contradiction. Hence this case cannot occur. (There may also be some 4-shares in \( (\frac{34}{68}, \frac{35}{68}) \) but this does not affect the argument.) \( \square \)

**Exercise 0.3.** Derive that the upper bound for \( f(29, 17) \) using the Half method is \( \frac{27}{68} \). \( (\text{Hint: See the paragraphs after the proof of Theorem 0.1}) \)