0.1 The Half method

On the slides we proved $f(11,5) \le \frac{13}{30}$. A brief review:

(1) Since $\frac{13}{30} > \frac{1}{3}$, every muffin is cut into 2 pieces, so there are $2m$ pieces.
(2) Since each muffin is cut into 2 pieces that are buddies, there are at most 11 pieces that are $> \frac{1}{2}$.
(3) We showed that any procedure with smallest piece $> \frac{13}{30}$ would have at least 12 shares $> \frac{1}{2}$. This gave a contradiction.

We generalize this technique, which we call *The Half method*. It works just as well if we end up with more than m shares $< \frac{1}{2}$.

There are many cases of the Half method. Therefore we give 2 more examples of what can happen when it is applied:

- $f(45,26) \le \frac{32}{78}$
- $f(29,17) \le \frac{27}{68}$

0.2 $f(45,26) \le \frac{32}{78}$ by The Half method

Theorem 0.1. $f(45,26) \le \frac{32}{78}$.

Proof. Assume, by way of contradiction, that there is a $(45,26)$-procedure with smallest piece $> \frac{32}{78}$. Since $\frac{32}{78} > \frac{1}{3}$ every muffin is cut into exactly 2 pieces. Hence there are 90 pieces. Note that there can be at most 45 pieces $< \frac{1}{2}$. We show that there is a piece $\le \frac{32}{78}$.

Every student gets $\frac{45}{26} = \frac{45\times3}{26\times3} = \frac{135}{78}$.

Case 1: Alice gets ≥ 5 shares. Then one of them is $< \frac{135}{78} \times \frac{1}{5} = \frac{27}{78} < \frac{32}{78}$.

Case 2: Bob gets ≤ 2 shares. Then one of the shares is $> \frac{135}{78} \times \frac{1}{2} = \frac{67.5}{78}$. Its buddy is $< 1 - \frac{67.5}{78} = \frac{10.5}{78} < \frac{32}{78}$.

In the subsequent cases we assume the negation of Cases 1 and 2. Hence everyone is either a 3-student or a 4-student. Let

s_3 (s_4) be the number of 3-students (4-students). Since there are 90 pieces and 26 students,

$$3s_3 + 4s_4 = 90$$
$$s_3 + s_4 = 26.$$

Hence $s_3 = 14$ and $s_4 = 12$. So there are fourteen 3-students, twelve 4-students, forty-two 3-shares, and forty-eight 4-shares. Since $48 > 45$, if all of the 4-shares are $< \frac{1}{2}$, that will be a contradiction. Indeed, this will be our contradiction.

We now look at intervals.

Case 3: Alice has a 4-share $\geq \frac{39}{78}$. Alice's other three 4-shares add up to $\leq \frac{135}{78} - \frac{39}{78} = \frac{96}{78}$, hence one of them is $\leq \frac{96}{78} \times \frac{1}{3} = \frac{32}{78}$.

Case 4: Bob has a 3-share $\leq \frac{43}{78}$. Bob's other two 3-shares add up to $\geq \frac{135}{78} - \frac{43}{78} = \frac{92}{78}$, hence one of the shares is $\geq \frac{92}{78} \times \frac{1}{2} = \frac{46}{78}$. Its buddy is $\leq 1 - \frac{46}{78} = \frac{32}{78}$.

Case 5: The following picture captures the negation of cases 1,2,3, and 4.

$$\begin{array}{ccccccc} (& 48\text{ 4-shs} &)[& 0 &](& 42\text{ 3-shs} &) \\ \frac{32}{78} & & \frac{39}{78} & & \frac{43}{78} & & \frac{46}{78} \end{array}$$

The midpoint is $\frac{1}{2} = \frac{39}{78}$. Note that all forty-eight 4-shares are $< \frac{1}{2}$. This is a contradiction. □

We show how one could *derive* the upper bound $f(45, 26) \leq \frac{32}{78}$. Let α be the upper bound. We derive conditions on α that will make the proof of $f(45, 26) \leq \alpha$ work. We assume $\alpha > \frac{1}{3}$. We guess everyone is either a 3-student or a 4-student.)

In the proof that $f(45, 26) \leq \frac{32}{78}$ we deduced that there are forty-two 3-shares and forty-eight 4-shares. This calculation *did not use that the goal was* $\frac{32}{78}$. Hence we can use that reasoning. We have the following picture, though we do not know x or y.

$$\begin{array}{llll} (\text{ 48 4-shs })[& 0 &](\text{ 42 3-shs } &) \\ \alpha & x & y & 1-\alpha \end{array}$$

What are x and y?

- x is the least number such that every 4-share is $< x$. Hence $3\alpha + x = \frac{135}{78}$, so $x = \frac{135}{78} - 3\alpha$.
- y is the largest number such that every 3-share is $> y$. Hence $2(1-\alpha) + y = \frac{135}{78}$, so $y = 2\alpha - \frac{7}{26}$.

Hence we have:

$$\begin{array}{llll} (\text{ 48 4-shs })[& 0 &](\text{ 42 3-shs } &) \\ \alpha & \frac{135}{78} - 3\alpha & 2\alpha - \frac{7}{26} & 1-\alpha \end{array}$$

If $x \le \frac{1}{2} \le y$ then there will be $48 > 45$ shares to the left of $\frac{1}{2}$ which is a contradiction. We look at setting $x = \frac{1}{2}$ and $y = \frac{1}{2}$.

If $x = \frac{1}{2}$ then

$$\alpha = \frac{\frac{135}{78} - \frac{1}{2}}{3} = \frac{16}{39}.$$

If $y = \frac{1}{2}$ then

$$\alpha = \frac{\frac{1}{2} + \frac{7}{26}}{2} = \frac{5}{13}.$$

You would think we should take the lower value, $\alpha = \frac{5}{13}$. But, alas, if you try to do the proof with this value you get that $y < x$ so the proof would not work. Hence we take $x = \frac{16}{39}$.

0.3 $f(29, 17) \le \frac{27}{68}$ by The Half method

In the proof of Theorem 0.1, the intervals containing the 3-shares and the intervals containing the 4-shares did not overlap. (This is the most common case for the Half method.) Is there a case where the intervals overlap and the Half method still works? Yes. We present one.

Theorem 0.2. $f(29, 17) \le \frac{27}{68}$.

Proof. Assume, by way of contradiction, that there is a $(29, 17)$-procedure with smallest piece $> \frac{27}{68}$. Since $\frac{27}{68} < \frac{1}{3}$ every muffin is cut into exactly 2 pieces. Hence there are 58 pieces. Note that there can be at most 29 pieces $> \frac{1}{2}$.

Every student gets $\frac{29}{17} = \frac{29\times4}{17\times4} = \frac{116}{68}$.

We leave as an exercise to show that (1) if Alice has ≥ 5 shares then she has a share $< \frac{27}{68}$, (2) if Bob has a ≤ 2 shares then one of them has a buddy that is $< \frac{27}{68}$, hence (3) everyone is a 3-student or a 4-student, and (4) there are ten 3-students, seven 4-students, thirty 3-shares, and twenty-eight 4-shares. Since $30 > 29$, if all of the 3-shares are $> \frac{1}{2}$, that will be a contradiction. Indeed, this will be our contradiction.

We now look at intervals.

Case 1: Alice has a 4-share $\ge \frac{35}{68}$. Alice's other three 4-shares sum to $\le \frac{116}{68} - \frac{35}{68} = \frac{81}{68}$, hence one of them is $\le \frac{81}{68} \times \frac{1}{3} = \frac{27}{68}$.

Case 2: Bob has a 3-share $\le \frac{34}{68}$. Bob's other two 3-shares sum to $\ge \frac{116}{68} - \frac{34}{68} = \frac{82}{68}$, hence one of the shares is $\ge \frac{82}{68} \times \frac{1}{2} = \frac{41}{68}$. Its buddy is $\le 1 - \frac{41}{68} = \frac{27}{68}$.

Case 3: The negation of cases 1 and 2. I know what you are thinking. We'll just draw the picture and have a good sense of what is going on. But the picture is hard to draw. Why? Let's draw the 4-share and 3-share pictures separately.

The 4-shares:

$$\begin{array}{ccc} (\text{ 28 4-shs } &)(\text{ 0 4-shs } &) \\ \frac{27}{68} & \frac{35}{68} & \frac{41}{68} \end{array}$$

The 3-shares:

$$\begin{array}{ccc} (\text{ 0 3-shs } &)(\text{ 30 3-shs } &) \\ \frac{27}{68} & \frac{34}{68} & \frac{41}{68} \end{array}$$

They overlap. The interval $(\frac{34}{68}, \frac{35}{68})$ can contain both 3-shares and 4-shares. Can our proof proceed anyway? Yes.

All thirty 3-shares are bigger than $\frac{1}{2}$. This is a contradiction. Hence this case cannot occur. (There may also be some 4-shares in $(\frac{34}{68}, \frac{35}{68})$ but this does not affect the argument.) □

Exercise 0.3. Derive that the upper bound for $f(29, 17)$ using the Half method is $\frac{27}{68}$. (*Hint:* See the paragraphs after the proof of Theorem 0.1.)