

The Muffin Problem

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How it Began

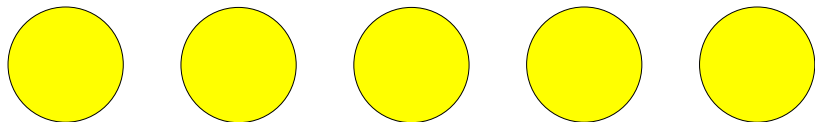
A Recreational Math Conference (Gathering for Gardner) May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival: A Sample of Mathematical Puzzles Compiled by Nancy Blachman

which had this problem, proposed by **Alan Frank**:

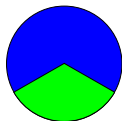
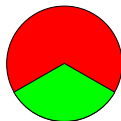
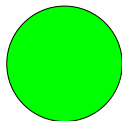
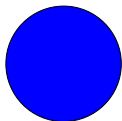
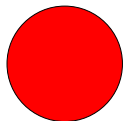
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



5 Muffins, 3 Students, Proc by Picture

Person	Color	What they Get
Alice	RED	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	BLUE	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

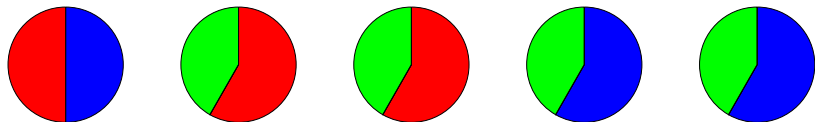
Work on it with your neighbor

5 Muffins, 3 People—Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	RED	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	BLUE	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor

5 Muffins, 3 People—Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N . We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$ -sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. (**Henceforth:** All muffins cut into ≥ 2 pieces.)

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins cut into 2 pieces.)

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \quad \text{Great to see } \frac{5}{12}$$

What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others

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There can't be much more to this.

If there is not much more to this then how come

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- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).

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The following happened:

- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).
- ▶ Come across a problem where the techniques do not work.

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The following happened:

- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).
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- ▶ Find a new technique **which was interesting**.

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The following happened:

- ▶ Find a technique that solves many problems (e.g., Floor-Ceiling).
- ▶ Come across a problem where the techniques do not work.
- ▶ Find a new technique **which was interesting**.
- ▶ Lather, Rinse, Repeat.

General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a **Mixed Int Program**.

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We have shown $f(m, s)$ exists, is rational, and is computable using a **Mixed Int Program**.

This was a case of a Theorem in **Applied Math** being used to prove a Theorem in **Pure Math**.

Amazing Results! / Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

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Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

Conventions

Duality Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

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We know and use the following:

1. By Duality Theorem can assume $m > s$
2. By REASONS we can assume m, s are relatively prime.
3. All muffins are cut in ≥ 2 pcs. Replace uncut muff with 2 $\frac{1}{2}$'s

7 Muffins, 3 Students

Work on $f(7, 3)$ in groups.

7 Muffins, 3 Students.

Get upper and lower bounds that match!

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We first look at LIMITS on what we can expect.

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Now what?

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6. That piece came from a muffin. Other piece is $\leq 1 - \frac{7}{12} = \frac{5}{12}$.

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7. Great! We know $f(7, 3) \leq \frac{5}{12}$.
8. Can we show a protocol that gives $f(7, 3) \geq \frac{5}{12}$?

7 Muffins, 3 Students: How to think about protocol

Want $f(7, 3) \geq \frac{5}{12}$.

7 Muffins, 3 Students: How to think about protocol

Want $f(7, 3) \geq \frac{5}{12}$.

Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.

7 Muffins, 3 Students: How to think about protocol

Want $f(7, 3) \geq \frac{5}{12}$.

Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.

Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.

Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.

Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.

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Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.

Need to know what combos of 5, 6, 7 add to 28.

$$7 + 7 + 7 + 7 = 28$$

$$5 + 5 + 6 + 6 + 6 = 28$$

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.
3. Give 1 student 4 pieces of size $\frac{7}{12}$.

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.

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$$7 + 7 + 7 + 7 = 28$$

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.
3. Give 1 student 4 pieces of size $\frac{7}{12}$.
4. Give 2 students 2 pieces of size $\frac{5}{12}$ and 3 pieces of size $\frac{6}{12}$.

8 Muffins, 3 Students

Work on $f(8, 3)$ in groups.

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Get upper and lower bounds that match!

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1. If a muffin is uncut, can cut it in two.
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3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.

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4. 3 students, so some student gets $\geq \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.

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4. 3 students, so some student gets $\geq \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.
5. 3 students, so some student gets $\leq \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\leq 1 - \frac{8}{15} = \frac{7}{15}$.

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3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.
4. 3 students, so some student gets $\geq \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.
5. 3 students, so some student gets $\leq \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\leq 1 - \frac{8}{15} = \frac{7}{15}$.
6. Great! We know $f(8, 3) \leq \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.

8 Muffins, 3 Students: How to think about it

We first look at LIMITS on what we can expect.

1. If a muffin is uncut, can cut it in two.
2. If a muffin is cut in ≥ 3 pieces then some piece $\leq \frac{1}{3}$. Unlikely that that's a good idea.
3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.
4. 3 students, so some student gets $\geq \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.
5. 3 students, so some student gets $\leq \lfloor \frac{16}{3} \rfloor = 5$ pieces. That student must get a piece $\geq \frac{8}{3} \times \frac{1}{5} = \frac{8}{15}$. So there is some piece of size $\leq 1 - \frac{8}{15} = \frac{7}{15}$.
6. Great! We know $f(8, 3) \leq \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.
7. Can we show a protocol that gives $f(8, 3) \geq \frac{4}{9}$?

8 Muffins, 3 Students: How to think about protocol

Want $f(8, 3) \geq \frac{4}{9}$.

8 Muffins, 3 Students: How to think about protocol

Want $f(8, 3) \geq \frac{4}{9}$.

Will be cutting some muffins $(\frac{4}{9}, \frac{5}{9})$.

8 Muffins, 3 Students: How to think about protocol

Want $f(8, 3) \geq \frac{4}{9}$.

Will be cutting some muffins $(\frac{4}{9}, \frac{5}{9})$.

$\frac{1}{2}$ was helpful last time so lets also include $\frac{4.5}{9}$.

Need to know what combos of $\frac{4}{9}, \frac{4.5}{9}, \frac{5}{9}$ add to $\frac{8}{3} = \frac{24}{9}$.

8 Muffins, 3 Students: How to think about protocol

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Need to know what combos of 4, 4.5, 5 add to 24

$$4 + 4 + 4 + 4 + 4 + 4 = 24$$

$$4.5 + 4.5 + 5 + 5 + 5 = 24$$

8 Muffins, 3 Students: How to think about protocol

Want $f(8, 3) \geq \frac{4}{9}$.

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$$4 + 4 + 4 + 4 + 4 + 4 = 24$$

$$4.5 + 4.5 + 5 + 5 + 5 = 24$$

1. Cut 6 muffins $(\frac{4}{9}, \frac{5}{9})$.
2. Cut 2 muffins $(\frac{4.5}{9}, \frac{4.5}{9})$.
3. Give 1 student six $\frac{4}{9}$ pieces.
4. Give 2 students two $\frac{4.5}{9}$ pieces and four $\frac{5}{9}$ pieces.

FC Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor}\right\}\right\}.$$

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Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

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Someone gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. \exists piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

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Someone gets $\leq \lfloor \frac{2m}{s} \rfloor$ pieces. \exists piece $\geq \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lfloor 2m/s \rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

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$$f(3k, 3) = 1.$$

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$$f(1, 3) = \frac{1}{3}$$

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$$f(3k + 1, 3) = \frac{3k-1}{6k}, k \geq 1.$$

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Note: A Mod 3 Pattern.

Theorem: For all $m \geq 3$, $f(m, 3) = \text{FC}(m, 3)$.

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$$f(4k, 4) = 1 \text{ (easy)}$$

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$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

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FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

$$f(4k + 1, 4) = \frac{4k-1}{8k}, k \geq 1.$$

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$$f(4k + 1, 4) = \frac{4k-1}{8k}, k \geq 1.$$

$$f(4k + 2, 4) = \frac{1}{2}.$$

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$$f(4k + 2, 4) = \frac{1}{2}.$$

$$f(4k + 3, 4) = \frac{4k+1}{8k+4}.$$

Note: A Mod 4 Pattern.

Theorem: For all $m \geq 4$, $f(m, 4) = \text{FC}(m, 4)$.

FOUR Students

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FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

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Note: A Mod 4 Pattern.

Theorem: For all $m \geq 4$, $f(m, 4) = \text{FC}(m, 4)$.

FC-Conjecture: For all m, s with $m \geq s$, $f(m, s) = \text{FC}(m, s)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

FIVE Students

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FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

FIVE Students

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FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

FIVE Students

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For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

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For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

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For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ **except $m=11$** , $f(m, 5) = \text{FC}(m, 5)$.

What About FIVE students, ELEVEN muffins?

$$f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}.$$

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.

We found a protocol with smallest piece $\frac{13}{30} = 0.4333\dots$

1. Divide 1 muffin $(\frac{15}{30}, \frac{15}{30})$.
2. Divide 2 muffins $(\frac{14}{30}, \frac{16}{30})$.
3. Divide 8 muffins $(\frac{13}{30}, \frac{17}{30})$.
4. Give 2 students $[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30}]$
5. Give 1 students $[\frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30}]$
6. Give 2 students $[\frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30}]$

So Now What?

We have:

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\dots$$

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Options:

1. $f(11, 5) = \frac{11}{25}$. Need to find procedure.
2. $f(11, 5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11, 5)$ in between. Need to find both.
4. $f(11, 5)$ unknown to science!

Vote

So Now What?

We have:

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\dots$$

Options:

1. $f(11, 5) = \frac{11}{25}$. Need to find procedure.
2. $f(11, 5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11, 5)$ in between. Need to find both.
4. $f(11, 5)$ unknown to science!

Vote WE SHOW: $f(11, 5) = \frac{13}{30}$. Exciting new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M .

The *other piece* from muffin M is the *buddy of x* .

Note that the buddy of x is of size

$$1 - x.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N . We want $N \leq \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N . We want $N \leq \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

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Case 3: Some student gets ≤ 3 pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

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One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
- ▶ s_5 is number of students who get 5 pieces

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
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$$4s_4 + 5s_5 = 22$$

$$s_4 + s_5 = 5$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
- ▶ s_5 is number of students who get 5 pieces

$$4s_4 + 5s_5 = 22$$

$$s_4 + s_5 = 5$$

$s_4 = 3$: There are 3 students who have 4 shares.

$s_5 = 2$: There are 2 students who have 5 shares.

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
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$$4s_4 + 5s_5 = 22$$

$$s_4 + s_5 = 5$$

$s_4 = 3$: There are 3 students who have 4 shares.

$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.

We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

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$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

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$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of z .

$$B(z) \leq 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of z .

$$B(z) \leq 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!

$$f(11, 5) = \frac{13}{30}, \text{ Fun Cases}$$

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares.
There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

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FC or HALF worked on everything with $s = 3, 4, 5, \dots, 23$.

Then we found a case where neither FC nor HALF worked.

We found a new method: INT.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$ -procedure with smallest piece $> \frac{19}{44}$.

Can assume all muffin cut in two and all student gets ≥ 2 shares.

We show that there is a piece $\leq \frac{19}{44}$.

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Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

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We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

Case 2: A student gets ≤ 3 shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$.

Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

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Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.

5-students: a student who gets 5 shares. s_5 is the number of them.

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$$4s_4 + 5s_5 = 48$$

$$s_4 + s_5 = 11$$

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$$4s_4 + 5s_5 = 48$$

$$s_4 + s_5 = 11$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.

$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

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Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

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Case 3.2: There is a share $\leq \frac{19}{44}$. Duh.
Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44} \right)$$

$$\left(\frac{19}{44}, \frac{25}{44} \right)$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

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5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

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Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

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Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

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Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

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$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

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Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

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Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} \\ \frac{20}{44} \end{array} \right) \left(\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

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Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

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4-share: a share that a 4-student who gets.

Claim: If some 4-share is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

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4-share: a share that a 4-student who gets.

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Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

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4-share: a share that a 4-student who gets.

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$$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

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$$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of z is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

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Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

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$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of z is of size

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Henceforth we assume all 4-shares are in

$$\left(\frac{21}{44}, \frac{25}{44} \right).$$

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

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Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.5: All Shares in Their Proper Intervals

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Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.

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$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 28 \text{ 4-shs} \\ \frac{21}{44} \end{array} \right) \left(\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

More Refined Picture of What is Going On

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Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

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If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$. QED.

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The following picture captures what we know so far.

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} \\ \frac{25}{44} \end{array} \right)$$

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S4= Small 4-shares

L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|L4|=20$.

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

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Claim 2: Every 4-student has at least 3 L4 shares.

Diagram

$$\binom{20 \text{ 5-shs}}{\frac{19}{44}} \binom{0}{\frac{20}{44}} \binom{8 \text{ S4-shs}}{\frac{21}{44}} \binom{0}{\frac{23}{44}} \binom{20 \text{ L4-shs}}{\frac{24}{44}} \binom{0}{\frac{25}{44}}$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \binom{23}{44} + 2 \times \binom{25}{44} = \frac{24}{11}.$$

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

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Contradiction: Each 4-student gets ≥ 3 L4 shares.

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There are $s_4 = 7$ 4-students.

Diagram

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Contradiction: Each 4-student gets ≥ 3 L4 shares.

There are $s_4 = 7$ 4-students.

Hence there are ≥ 21 L4-shares.

Diagram

$$\left(\begin{array}{c} 20 \text{ 5-shs} \\ \frac{19}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{20}{44} \end{array} \right] \left(\begin{array}{c} 8 \text{ S4-shs} \\ \frac{21}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{23}{44} \end{array} \right] \left(\begin{array}{c} 20 \text{ L4-shs} \\ \frac{24}{44} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{25}{44} \end{array} \right]$$

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Contradiction: Each 4-student gets ≥ 3 L4 shares.

There are $s_4 = 7$ 4-students.

Hence there are ≥ 21 L4-shares. But there are only 20.

Other Techniques

Here are the list of the Techniques we came up with:

1. Floor-Ceiling
2. Half
3. INT
4. GAP
5. Easy buddy-match
6. Hard buddy-match
7. Train (only worked on 3 (m, s) 's).

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Time to say we are NOT going to find a finite set of techniques that covers all cases and take what we got and write a book.

Later Results by Other People

1. In Fall 2018 Scott Huddleston emailed me code for an algorithm that, on input m, s , found $f(m, s)$ REALLY FAST.
2. Jacob and Erik Understand WHAT his algorithm does and Jacob coded it up to make sure he understood it. Jacob's code is also REALLY FAST.
3. Neither Scott, Bill, Jacob, or Erik had a proof that Scott's algorithm was fast (poly in m, s).
4. Richard Chatwin independently came up with the same algorithm; however, he also has a proof that it works. Its on arXiv.
5. One corollary of the work: $f(m, s)$ only depends on m/s .

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We also all could do some of the other tasks.

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Team Work Makes the Dream Work!