The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
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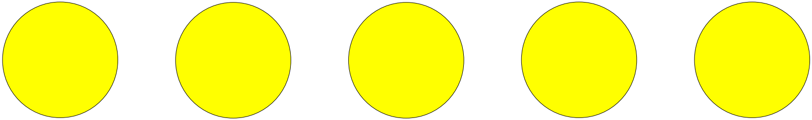
I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

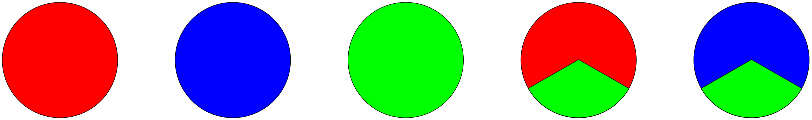
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ *where nobody gets a tiny sliver?*



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1+\frac{2}{3}=\frac{5}{3}$
Bob	**BLUE**	$1+\frac{2}{3}=\frac{5}{3}$
Carol	GREEN	$1+\frac{1}{3}+\frac{1}{3}=\frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

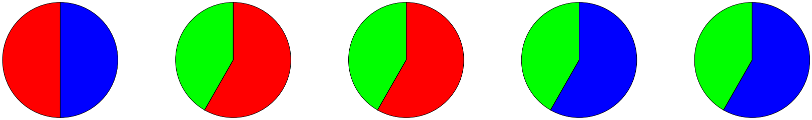
Is there a procedure with a larger smallest piece?

YES WE CAN!

Five Muffins, Three People–Proc by Picture

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Carol	GREEN	$\frac{5}{12}+\frac{5}{12}+\frac{5}{12}+\frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

NO WE CAN'T!

Five Muffins, Three People–Can't Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \le \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.
(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

General Problem

$f(m,s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3) = \frac{5}{12}$ here.

We have shown $f(m,s)$ exists, is rational, and is computable using a Mixed Int Program (in paper).

Amazing Results!/Amazing Theorems!

1. $f(43,33) = \frac{91}{264}$.
2. $f(52,11) = \frac{83}{176}$.
3. $f(35,13) = \frac{64}{143}$.

All done by hand, no use of a computer
by Co-author Erik Metz is a muffin savant !

Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
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3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?
NO

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by 3/5.

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by 3/5.

Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.

Floor-Ceiling Thm (FC Thm) Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le FC(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\le \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \mathbf{\frac{m}{s\lceil 2m/s\rceil}}$.

Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1.$

$f(3k+1,3) = \frac{3k-1}{6k},\ k \geq 1.$

$f(3k+2,3) = \frac{3k+2}{6k+6}.$

Note: A Mod 3 Pattern.
Theorem: For all $m \geq 3$, $f(m,3) = FC(m,3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$f(4k, 4) = 1$ (easy)

$f(1, 4) = \frac{1}{4}$ (easy)

$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = FC(m, 4)$.
FC-Conjecture: For all m, s with $m \geq s$, $f(m, s) = FC(m, s)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k+2, 5) = \frac{5k-2}{10k}$. $f(7,5) = FC(7,5) = \frac{1}{3}$

For $k \geq 1$, $f(5k+3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k+4, 5) = \frac{5k+1}{10k+5}$
Note: A Mod 5 Pattern.
Theorem: For all $m \geq 5$ **except m=11**, $f(m, 5) = FC(m, 5)$.

What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11,5) \ge \frac{13}{30}$.
2. $f(11,5) \le \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5\rceil}, 1 - \frac{11}{5\lfloor 22/5\rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

If $f(5,11) < \frac{11}{25}$ then FC-conjecture is false!

What About FIVE students, ELEVEN muffins?

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So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

If $f(5,11) < \frac{11}{25}$ then FC-conjecture is false!

$$\text{WE SHOW: } \mathbf{f(11,5) = \frac{13}{30}}$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: is $\le \frac{1}{2}$. Then there is a piece

$$\ge \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.$$

The other piece from the muffin is

$$\le 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.$$

Case 4.2: All 4-shares are $> \frac{1}{2}$. So there are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

Essence of the Interval Method

1. Every muffin cut into two pieces.
2. Find L such that some students get either L or $L+1$ pieces.
3. Find how many students get L $(L+1)$ pieces.
4. Find intervals that these pieces must be in.
5. Find how many pieces are in an interval
6. Get a contradiction out of this.

Note: Can turn Interval Theorem into a function INT such that $f(m,s) \leq INT(m,s)$.

FC CONJECTURE STILL SORT OF TRUE

FC Conj: For all $m \geq s$, $f(m,s) = FC(m,s)$. FALSE

Theorem: For fixed s, for $m \geq \frac{s^3+2s^2+s}{2}$ $f(m,s) = FC(m,s)$.

Statistics: For $3 \leq s \leq 50$, $s+1 \leq m \leq 59$:

$f(m,s) = FC(m,s)$ in 683 cases
$f(m,s) = INT(m,s)$ in 194 cases

Still 108 cases left. Need new technique!

The Buddy-Match Method! (BM)

Can FC and INT do everything?
No.
They are very good when $\frac{2m}{s} > 3$ but NOT so good otherwise.
We do a concrete example of **The Buddy-Match Method**

$$f(43,39) \le \frac{53}{156}$$

(We have matching lower bound also)

Definition: Assume we have a protocol where all students get 2 or 3 shares. If x is a 2-share then the other share that student has is the shares **match**. Note that $M(x) = \frac{m}{s} - x$.
Warning: We will apply M to intervals. These intervals have to have only 2-shares in them! But they will!

$f(43,39) \le \frac{53}{156}$

Theorem $f(43,39) \le \frac{53}{156}$ ($\ge$ also known).
Assume there is an $(43,39)$-procedure with smallest piece $> \frac{53}{156}$.
Can assume all muffins cut in 2 pieces, all students get ≥ 2 shares.

Case 1: A student gets ≥ 4 shares. Some share $\le \frac{43}{39\times 4} < \frac{53}{156}$.

Case 2: A student gets ≤ 1 shares. Can't occur.

Case 3: Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86.

How Many Students get Two Shares? Three Shares?

Let s_2 (s_3) be the number of 2-students (3-students).

$$\begin{aligned} 2s_2 + 3s_3 &= 86 \\ s_2 + s_3 &= 39 \text{ Get } s_2 = 31 \text{ and } s_3 = 8 \end{aligned}$$

Case 3.1, 3.2, 3.3, 3.4:
$(\exists)$ 3-share $\geq \frac{66}{156}$. Rm. Now 2-shares $\geq \frac{43}{39} - \frac{66}{156} = \frac{53}{78}$.

So some share $\leq \frac{53}{156}$.
By similar reasoning (Case 3.2, 3.3, 3.4) we have:

(	24 3-shs	)[	0 shs	](	62 2-shs	)
$\frac{53}{156}$		$\frac{66}{156}$		$\frac{69}{156}$		$\frac{103}{156}$

The Buddy-Match Method

$$\underset{\frac{53}{156}}{(} \quad 24 \text{ 3-shs} \quad \underset{\frac{66}{156}}{)[} \quad 0 \text{ shs} \quad \underset{\frac{69}{156}}{](} \quad 62 \text{ 2-shs} \quad \underset{\frac{103}{156}}{)}$$

$$|(\tfrac{53}{156}, \tfrac{69}{156})| = 24$$

$$|B(\tfrac{53}{156}, \tfrac{69}{156})| = |\tfrac{87}{156}, \tfrac{103}{156}| = 24$$

$$|M(\tfrac{87}{156}, \tfrac{103}{156})| = |\tfrac{69}{156}, \tfrac{85}{156}| = 24$$

$$|(\tfrac{53}{156}, \tfrac{69}{156}) \cup (\tfrac{69}{156}, \tfrac{85}{156}) \cup (\tfrac{87}{156}, \tfrac{103}{156})| = 24 \times 3 = 72$$

$$|(\tfrac{85}{156}, \tfrac{87}{156})| = 86 - 72 = 14.$$

More Buddy-Match Method

$|(\frac{85}{156}, \frac{87}{156})| = 14$. Buddy-Match yields $|(\frac{53}{156}, \frac{55}{156})| = 14$

$|[\frac{66}{156}, \frac{69}{156}]| = 0$. Buddy-Match yields $|[\frac{55}{156}, \frac{58}{156}]| = 0$.

The following picture captures what we know so far about 3-shares.

$$\begin{array}{ccccccc} (& 14 &)[& 0 &](& 10 &) \\ \frac{53}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{66}{156} \end{array}$$

Big Shares and Small Shares

$$\begin{array}{ccccccc} (& 14 &)[& 0 &](& 10 &) \\ \frac{53}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{66}{156} \end{array}$$

- Shares in $(\frac{53}{156}, \frac{55}{156})$ are *small shares*;
- Shares in $(\frac{58}{156}, \frac{66}{156})$ are *large shares*;

Notation d_i is numb of students who have i small shares ($3 - i$ large shares).

$d_0 = 0$ since $3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39}$.

$d_3 = 0$ since $3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39}$.

SO there are NO d_0-students or d_3-students.

d_1 and d_2 Students Cause a Gap!

$$\underset{\frac{53}{156}}{(} \quad 14 \quad \underset{\frac{55}{156}}{)[} \quad 0 \quad \underset{\frac{58}{156}}{](} \quad 10 \quad \underset{\frac{66}{156}}{)}$$

d_1: If a d_1-student has a large shares $\ge \frac{61}{156}$ then he will have

$$> \frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}.$$

Upshot: Large shares of d_1-student are in $(\frac{58}{156}, \frac{61}{156})$.

d_2: If a d_2-student has a large shares $\le \frac{62}{156}$ then he will have

$$< \frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}.$$

Upshot: Large shares of a d_2-student are in $(\frac{62}{156}, \frac{66}{156})$.

Upshot Upshot: There are NO shares in $[\frac{61}{156}, \frac{62}{156}]$

Even More Buddy Match

The following picture captures what we know so far about 3-shares.

(	14	)[	0	](	x	)[	0	](	y	)
$\frac{53}{156}$		$\frac{55}{156}$		$\frac{58}{156}$		$\frac{61}{156}$		$\frac{62}{156}$		$\frac{66}{156}$

Use Buddy-Match to show that $|(\frac{61}{156}, \frac{62}{156})| = |(\frac{62}{156}, \frac{63}{156})|$. So:

(	14	)[	0	](	x	)[	0	](	y	)
$\frac{53}{156}$		$\frac{55}{156}$		$\frac{58}{156}$		$\frac{61}{156}$		$\frac{63}{156}$		$\frac{66}{156}$

$$x + y = 10.$$

Use Buddy-Match to show that $|(\frac{58}{156}, \frac{61}{156})| = |(\frac{63}{156}, \frac{66}{156})|$ so they are are both 5.

(	14	)[	0	](	5	)[	0	](	5	)
$\frac{53}{156}$		$\frac{55}{156}$		$\frac{58}{156}$		$\frac{61}{156}$		$\frac{63}{156}$		$\frac{66}{156}$

Equations

$$\underset{\frac{53}{156}}{(} \quad 14 \quad \underset{\frac{55}{156}}{)[} \quad 0 \quad \underset{\frac{58}{156}}{](} \quad 5 \quad \underset{\frac{61}{156}}{)[} \quad 0 \quad \underset{\frac{63}{156}}{](} \quad 5 \quad \underset{\frac{66}{156}}{)}$$

Only the d_2-students use $(\frac{63}{156}, \frac{66}{156})$. Every d_2 student uses one share from that interval:

$$d_2 = 5.$$

Each d_i student uses i shares from $(\frac{53}{156}, \frac{55}{156})$:

$$1 \times d_1 + 2 \times d_2 = 14: \text{ So } d_1 = 4$$

There are 8 3-students:

$$d_1 + d_2 = 8: \text{ So } 5 + 4 = 8.\text{CONTRADICTION!}$$

The Essence of The Buddy-Match Method

1. Works when $\lceil \frac{2m}{s} \rceil = 3$: Just 2-shares and 3-shares.
2. $2m$ pieces, s_2 students get 2 shares, s_3 students get 3 shares.
3. Find a GAP
4. Using BM Sequence on 3-shares-interval find intervals that cover **almost** the entire interval. Missing an interval (a, b).
5. Use BM on (a, b) to get info on an initial interval of 3-shares.
6. Use BM on GAP to get GAPs within the 3-shares.
7. Set up linear equations relating intervals and types of students.
8. Show that system has no solution in $\mathbf{N}$.

Note: Can turn BM technique into a function $BM(m, s)$ such that $f(m, s) \leq BM(m, s)$.

Statistics

For $3 \leq s \leq 60$, $s+1 \leq m \leq 70$, m, s rel prime:

$f(m,s) = FC(m,s)$ in 927 cases. $\sim 68\%$
$f(m,s) = INT(m,s)$ in 268 cases. $\sim 20\%$
$f(m,s) = BM(m,s)$ in 85 cases. $\sim 6\%$
$f(m,s) = ERIK(m,s)$ in 80 cases. $\sim 6\%$
All cases solved!

A Guess that Works. But Why?

1) We suspected there was a constant X such that:

$$(\forall k \geq 1)\left[f(21k+11, 21k+4) \leq \frac{7k+X}{21k+4}\right]$$

2) We knew that $f(11,4) = \frac{9}{20}$ so we conjectured $X = \frac{9}{5}$.

3) We prove the result with $X = \frac{9}{5}$ and $k \geq 1$ using BM. We prove matching lower bound for several k.

4) But the proof for $f(11,4)$ $(k=0)$ **cannot use BM and is totally unrelated to the proof for $k \geq 1$.**
Note: This technique always worked!

Another Guess that Works But we Don't Know Why

Want to know $f(41, 19)$. Can't use BM.
$41 - 19 = 22$. So try to prove, diff d is always Mod $3d$ pattern.
Need X:

$$(\forall k \geq 1)\left[f(66k + 41, 66k + 19) \leq \frac{22k + X}{66k + 19}\right]$$

Find X using BM and linear algebra (have program for that).
Get conj: $f(41, 19) = \frac{X}{19}$.
Note: This seems to always work but have not been able to use to get new results yet.

Programs

We have a program that on input (m, s):

1. We we used FC, INT, BM to get upper bounds.
2. BM method is a theorem generator.
3. Use linear algebra to try to find a lower bound (a procedure).

Results

1. FC, INT, and BM upper bounds on $f(m, s)$
2. For fixed s, for $m \geq\sim s^3$, $f(m, s) = FC(m, s)$.
3. For all $m \geq s$ $f(m, s) \geq \frac{1}{3}$.
4. For $1 \leq s \leq 7$ have proven formulas for $f(m, s)$. Mod s pattern
5. For $s = 8, \ldots, 100$ conjectures for $f(m, s)$. $f(m, s)$ seems to be a mod s pattern.
6. For $1 \leq d \leq 7$ have proven formulas for $f(s + d, s)$. Mod $3d$ pattern.
7. For all d conjecture that our Theorem Generator gives $f(s + d, s)$.
8. Conjecture that for all a, d there exists X such that

$$(\forall k \geq 0)\left[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a}\right]$$

Later Results by Other People

1. In Fall 2018 Scott Huddleston had code for an algorithm that, on input m, s, found $f(m, s)$ and the procedure REALLY FAST.
2. Jacob and Erik Understand WHAT his algorithm does and Jacob coded it up to make sure he understood it. Jacob's code is also REALLY FAST.
3. Neither Scott, Bill, Jacob, or Erik had a proof that Scott's algorithm was fast (poly in m, s).
4. Richard Chatwin independently came up with the same algorithm; however, he also has a proof that it works. Its on arixv.
5. One corollary of the work: $f(m, s)$ only depends on m/s.

Accomplishment I Am Most Proud of

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Convinced

- 4 High School students (Guang, Naveen, Naveen, Sunny)
- 3 college student (Erik, Jacob, Daniel)
- 1 professor (John D)

that the most important field of Mathematics is **Muffinry.**